INFINITE DIMENSIONAL UNIVERSAL BURGER-MOZES GROUPS

RÉMI BARRITAULT

Abstract. Given a countably infinite first-order structure $\mathcal{M}$ and a suitable coloring of the regular tree of countably infinite valency $\mathbf{T}$, one can define the subgroup $\mathrm{Aut}_{\mathcal{M}}(\mathbf{T})$ of $\mathrm{Aut}(\mathbf{T})$ consisting of all the automorphisms of $\mathbf{T}$ with local actions prescribed by $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathcal{M}$. This is a generalization of a construction carried out by Burger and Mozes in the finite valency case.

Under some transitivity and mild model-theoretic hypotheses on $\mathfrak{M}$, we fully classify the continuous irreducible unitary representation of $\mathrm{Aut}_{\mathcal{M}}(\mathbf{T})$. This is achieved by building on a similar classification by Ol'šanskiĭ for the whole group $\mathrm{Aut}(\mathbf{T})$, which can be seen as the special case where $\mathfrak{M}$ is the trivial structure. This provides new examples of locally Roelcke-precompact Polish groups of Type I.

Introduction

In abstract harmonic analysis, the notion of *Type I*, a very general property taking roots in the study of von Neumann algebras, forms a dividing line between topological groups whose representation theory is tractable and those for which it is too wild to be fully understood. We refer to [Bd20] for a formal definition and only give an illustration: by a celebrated theorem of Glimm [Gli61], a locally compact group G is of Type I if and only if every continuous unitary representation of G decomposes *uniquely* as a direct integral of irreducible representations. For such reasons, deciding if a given topological group G is of Type I or not is a crucial step in studying G from the prism of its representation theory.

For fundamental reasons, all abelian groups are of Type I, even as discrete groups. In fact, a discrete group is of Type I if and only if it is virtually abelian by a classical result of Thoma [Tho68]. In the non-discrete case, the picture is a lot richer while remaining, to this day, far less complete. Since locally compact groups come with natural operator algebras associated to them, most of the effort in the study of Type I groups has been devoted to the locally compact case. First, compact groups are of Type I and, more generally, all locally compact groups are determined by their irreducible unitary representation by the Gel'fand-Raikov Theorem. In particular, they all

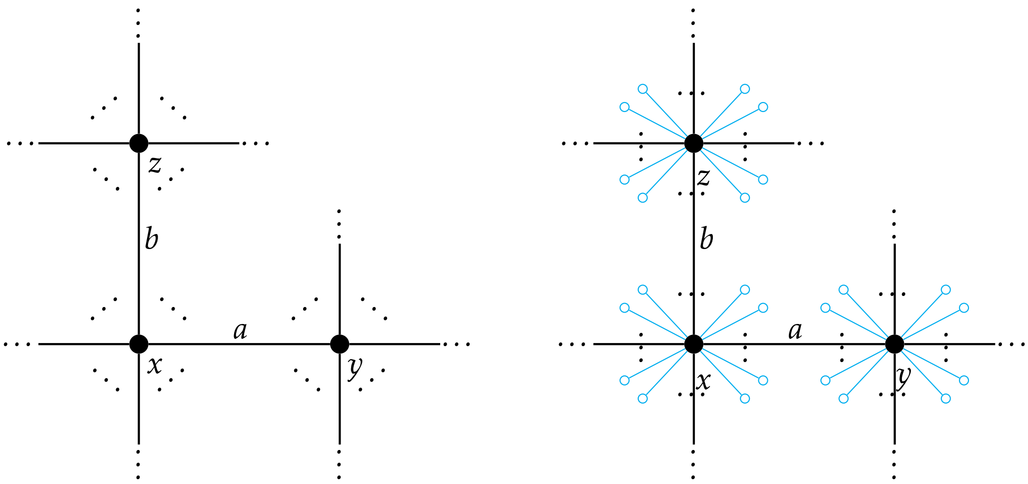

Figure 1. The colored trees $\mathbf{T}$ and $\mathbf{T}^{\mathrm{eq}}$.

have *faithful* representations. Examples of Type I locally group include all of the connected semi-simple [Har54] or nilpotent Lie groups [Dix59, Kir61] and all connected real algebraic groups [Dix57]. On the other hand, the Mautner group is a solvable connected Lie group which is not of Type I (due to Mautner, unpublished, see [Bag78]).

At the opposite end of connected Lie groups are the totally disconnected locally compact groups, for which a large part of the landscape is occupied by those who act on trees. In 1999, Nebbia obtained a striking result linking the regularity of the representation theory of such a group to the transitivity of its action on the tree (the *CCR property* is a strengthening of the Type I property):

Theorem ([Neb99]). *Let T_d be the regular tree of finite valency $d \in \mathbb{N}_{\geqslant 3}$ and let G be a closed unimodular subgroup of $\mathrm{Aut}(T_d)$ acting transitively on T_d.*
If G is CCR, then G acts transitively on the boundary of X.

In the same paper, Nebbia conjectured that the converse also holds, i.e. that having a very transitive action is the same as having a very tame representation theory for such groups. Nebbia's CCR conjecture is still open in its full generality but some substantial progress has been made recently. Houdayer and Raum provided a strong positive answer for non-amenable groups: a transitive non-amenable closed subgroup of $\mathrm{Aut}(\mathbf{T}_d)$ that does not act transitively on the boundary is not even of Type I [HR19].

Since then, stronger and more general results have been obtained [CKM23] but Houdayer and Raum's results also fully answer Nebbia's CCR conjecture in the positive for a particularly interesting class of groups: the *Burger-Mozes universal groups*. Given T_d the regular tree of finite valency $d \in \mathbb{N}_{\geqslant 3}$, one can color the unoriented edges of T_d with labels in $\{1,\ldots,d\}$. Then, every pair (g,x) where g is an automorphism of T_d and x is a vertex of T_d induces a bijection of $\{1,\ldots,d\}$ by keeping track of how g sends the edges around x to the edges around $g(x)$. This bijection is called the *local action* of g at x. Now, given a group $F \leqslant \mathrm{Sym}(\{1,\ldots,d\})$, one can consider the subgroup $U(F)$ of $\mathrm{Aut}(T_d)$ consisting of all the automorphisms g of T_d such that for every vertex x in T_d, the local action of g at x is given by an element of F. See below for a more formal definition in the infinite dimensional case. Houdayer and Raum proved the following:

Theorem ([HR19]). *Let T_d be the regular tree of finite valency $d \in \mathbb{N}_{\geqslant 3}$ and let $F \leqslant \mathrm{Sym}(\{1,\ldots,d\})$. The following properties are equivalent:*

- *(i) $U(F)$ is of Type I,*
- *(ii) $U(F)$ is CCR,*
- *(iii) the action $F \curvearrowright \{1,\ldots d\}$ is 2-transitive,*
- *(iv) the action of $U(F)$ on the boundary of T_d is transitive.*

As for the present paper, we will be studying an infinite dimensional analogue of these groups, which will be non-locally compact *Polish groups*. Leaving the locally compact world allows for much more pathological examples, going against most of the theory developed in the classical setting: for example, $L^2([0,1],\mathbb{S}_1)$ has a faithful representation but no non-trivial irreducible representation (hence is not of Type I) [Gla98] while $\mathrm{Homeo}_+([0,1])$ has no non-trivial representation, not even on a reflexive Banach space [Meg01] (hence is of Type I for trivial reasons).

Still, the representations of such groups have been studied at least since the 1970's. One of the first examples of Type I non-locally compact groups is the infinite symmetric group $\mathcal{S}_\infty$ of all permutations of a countably infinite set [Lie72]. A few years later, Ol'šanskiĭ developed the *semigroup* method and used it to find Type I groups by classifying all the unitary representations of a variety of groups such as the isometry group of the separable (real or complex) Hilbert space, the symplectic variations of it, as well as some infinite dimensional general linear groups over finite fields (see the survey [Ol'91]). More closely related to our work, Ol'šanskiĭ also classified the irreducible representations of $\mathrm{Aut}(T)$ where T is a regular tree with *infinite* valency [Ol'80]. Suppose G acts on $\mathbf{T}$. Given $X \subseteq \mathbf{T}$, G_X will denote the *pointwise stabilizer* of X while $\widetilde{G}_X$ will denote the *setwise stabilizer* of X. A continuous irreducible unitary representation of G is *non-spherical* if it admits no non-zero vector that is invariant under the action of a vertex stabilizer.

For the notion of induced representation, we refer to [Bd20]. The part of Ol'šanskiĭ's classification we will be focusing on is the following:

Theorem ([Ol'80])**.** *Let* $\mathbf{T}$ *be the regular tree with countably infinite valency and let* $G = \mathrm{Aut}(\mathbf{T})$*. The non-spherical continuous irreducible unitary representations of* $\mathrm{Aut}(\mathbf{T})$ *are exactly the representations of the form*

$$\mathrm{Ind}_{\widetilde{G}_X}^{G}(\sigma)$$

where $X \subseteq \mathbf{T}$ *is a finite subtree and* σ *is an irreducible representation of the finite group* $\widetilde{G}_X/G_X$*.*

The representation theory of non-locally compact Polish groups has recently started to draw more attention, in connection with model theory and descriptive set theory. Crucial to our approach will be Tsankov's full classification of the unitary representations of *Roelcke-precompact non-archimedean Polish groups* [Tsa12]. This result, which encompasses Lieberman's and most of Ol'šanskiĭ's, shows that Roelcke-precompact Polish groups behave similarly to compact groups and are of Type I. This applies in particular to automorphism groups of $\aleph_0$-categorical structures. Another classification theorem in the same flavor has been obtained in [BJJ24]. This adds non-Roelcke-precompact examples to the list, such as the isometry groups of Urysohn spaces, which also are of Type I. These results motivate the study of *infinite dimensional* universal Burger-Mozes groups. Indeed, one can carry out a similar construction as the one described above by replacing T_d with $\mathbf{T}$, the regular tree with countably infinite valency, and $F \curvearrowright \{1,\dots,d\}$ with $\mathrm{Aut}(\mathfrak{M}) \curvearrowright \mathfrak{M}$ where $\mathfrak{M}$ is a given countably infinite first-order structure. We will be denoting by $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ the closed subgroup of $\mathrm{Aut}(\mathbf{T})$ arising in this manner (a precise definition is given below). Another infinite dimensional generalization of Burger-Mozes groups has already been introduced by Smith [Smi17]. Using a different formalism that does not involve model theory, Smith studied the structure and dynamics on bi-regular trees with a focus on locally compact instances of the construction.

Back to our context and focusing on unitary representations, if $\mathfrak{M}$ is such that the results from either [Tsa24] or [BJJ24] apply, we can obtain a classification similar to Ol'šanskiĭ's under some additional hypotheses. On the side of [BJJ24], here is one way to make things work: We will say that $\mathfrak{M}$ has *finite algebraicity* if $\mathrm{acl}_{\mathfrak{M}}(A)$ is finite for every finite $A \subseteq \mathfrak{M}$ and that $\mathfrak{M}$ *weakly eliminates imaginaries* if for every open subgroup $U \leqslant \Gamma := \mathrm{Aut}(\mathfrak{M})$, there exists a finite subset $A \subseteq \mathfrak{M}$ such that:

$$\Gamma_A \leqslant U \leqslant \widetilde{\Gamma}_A\,.$$

Slightly extending the formalism of [BJJ24], we will say that $\mathfrak{M}$ is *dissociated* if for every unitary representation $\pi\colon \Gamma \curvearrowright \mathcal{H}$ and all *algebraically closed* subsets $A, B \subseteq \mathfrak{M}$,

$$\mathcal{H}_A \perp_{\mathcal{H}_{A\cap B}} \mathcal{H}_B$$

where $\mathcal{H}_A$ is the subspace of Γ_A invariant vectors.

The following is a direct generalization of Ol'šanskiĭ's classification for $\mathrm{Aut}(\mathbf{T})$ and can be proved directly with almost the same proof.

Theorem 1. *Let* $G := \mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ *where* $\mathrm{Aut}(\mathfrak{M}) \curvearrowright \mathfrak{M}$ *is 2-transitive and* $\mathfrak{M}$ *is dissociated with finite algebraicity.*

The non-spherical continuous irreducible unitary representations of G are exactly the representations of the form

$$\mathrm{Ind}_{\widetilde{G}_X}^{G}(\sigma)$$

where $X \subseteq \mathbf{T}$ *is a finite subtree that is algebraically closed with respect to the action* $G \curvearrowright \mathbf{T}$ *and* σ *is an irreducible representation of the group* $\widetilde{G}_X/G_X$*.*

This paper is devoted to proving a similar classification theorem that allows structures which do not weakly eliminates imaginaries. Namely, we will work with $\aleph_0$ categorical structures and Tsankov's classification, assuming that $\mathrm{Aut}(\mathfrak{M}) \curvearrowright \mathfrak{M}$ is *primitive* and has *no algebraicity over singletons*, not even in $\mathfrak{M}^{\mathrm{eq}}$ (i.e. such that every $a \in \mathfrak{M}$, $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a)$). See Theorem 2.22 for the full statement. Examples to which this applies is $\mathfrak{M} = \mathbb{P}^{\infty}(q)$, the separable infinite dimensional projective space on the finite field $\mathbb{F}_q$, where q is a power of a prime number. $\mathfrak{M}$ can also be an $\aleph_0$-categorical Fraïssé limit, provided that one is careful enough with the relations and functions at low arity.

The rest of the irreducible representations of $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$, namely the *spherical* ones, are independent of $\mathfrak{M}$, provided that one assumes 2-transitivity in accordance with Nebbia's CCR conjecture. In this case, Ol'šanskiĭ's classification of the spherical representations of $\mathrm{Aut}(\mathbf{T})$ [Ol'80, Sec. 5.3–5.6] applies as is to the subgroup $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ and just as $\mathrm{Aut}(\mathbf{T})$ is a Type I group, we obtain the following:

Corollary 1. *Let $\mathfrak{M}$ be a 2-transitive first-order structure, either as in Theorem 1 or both $\aleph_0$-categorical and such that:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

The group $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ *is of Type I.*

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1. Preliminaries

1.1. **Colored trees.**

Trees, i.e. acyclic connected graphs, will always be identified with the countable set of their vertices. In other words "$x \in X$" where X is a tree will always mean that x is a vertex of X. Let X be a tree and let $x, y \in X$. The set of vertices of X *adjacent* to x, called the *neighbors* of x, will be denoted $V_X(x)$. We will write $x \sim y$ if x and y are adjacent vertices of a tree. If x has exactly one neighbor in X, it will be called a *leaf*. We will denote by $E(X)$ the set of *unoriented edges* of X, i.e. the set of pairs $\{x, y\}$ of adjacent vertices in X, and we will write $E_X(x)$ for the set of all edges of X that involve x.

A *path* from x to y is a tuple $(x_0, \dots, x_n)$ of vertices of X such that $x_0 = x,\ x_n = y$ and $x_i \sim x_{i+1}$ for every $i < n$. In this setting, $n \in \mathbb{N}$ is the *length* of this path. Trees naturally come with a *metric structure*. Indeed, there exists a unique *shortest path* $[x, y]$ from x to y and we can define $d(x, y)$ to be the length of $[x, y]$. Then (X, d) is a metric space and $\mathrm{Aut}(X) = \mathrm{Iso}(X, d)$. We will write, for every $n \in \mathbb{N}$:

$$B_X(x, n) = \{z \in X,\ d(x, z) \leqslant n\} \qquad \text{and} \qquad S_X(x, n) = \{z \in X,\ d(x, z) = n\}.$$

For a subset or tuple $E \subseteq X$, $\langle E \rangle$ will denote the subtree of X generated by E, meaning:

$$\langle E \rangle = \bigcup \{[z, z'],\ z, z' \in E\}.$$

If x and y are distinct, we will adopt the following notation for the *cone* arising from x to y:

$$\mathrm{Cone}_x(y) := \{z \in X,\ y \in [x, z]\}.$$

$\mathbf{T}$ will denote the *regular tree with countable infinite valency*. Given a countably infinite first-order structure $\mathfrak{M}$, an $\mathfrak{M}$-*coloring* of $\mathbf{T}$ is a labeling of its unoriented edges by elements of $\mathfrak{M}$, i.e. a map $E(\mathbf{T}) \longrightarrow \mathfrak{M}$, such that for every $x \in \mathbf{T}$, the restriction $c\colon E_{\mathbf{T}}(x) \longrightarrow \mathfrak{M}$ is a bijection. A back and forth argument shows that there exists a unique $\mathfrak{M}$-coloring of $\mathbf{T}$ up to conjugation by an element of $\mathrm{Aut}(\mathbf{T})$. From now on, we thus fix such a structure $\mathfrak{M}$, an $\mathfrak{M}$-coloring c and still denote by $\mathbf{T}$ the resulting colored tree. Any additional hypothesis on $\mathfrak{M}$ will be specified in each result separately. The colored tree $\mathbf{T}$ is represented in Figure 1 (left) where $x, y, z \in \mathbf{T},\ a, b \in \mathfrak{M}$ and $c(x, y) = a,\ c(x, z) = b$.

Given $g \in \mathrm{Aut}(\mathbf{T})$ and $x \in \mathbf{T}$, the coloring c allows us to talk about the *local action g_x of g at x*. Indeed, let g_x be the unique bijection of $\mathfrak{M}$ such that the following diagram commutes:

$$\begin{tikzcd} E_{\mathbf{T}}(x) \arrow[r, "g"] \arrow[d, "c"'] & E_{\mathbf{T}}(g(x)) \arrow[d, "c"] \\ \mathfrak{M} \arrow[r, "g_x"] & \mathfrak{M} \end{tikzcd}$$

Fixing $x \in \mathbf{T}$, the map $g \longmapsto g_x$ is a *cocycle*:

$$\forall g, h \in \mathrm{Aut}(\mathbf{T}),\ (gh)_x = g_{h(x)} \circ h_x\ .$$

Definition 1.1. Let $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ be the following closed subgroup of $\mathrm{Aut}(\mathbf{T})$:

$$\{g \in \mathrm{Aut}(\mathbf{T}), \forall x \in \mathbf{T},\ g_x \in \mathrm{Aut}(\mathfrak{M})\}.$$

Remark 1.2. This group clearly depends on the choice of the coloring c. However, if c' is another $\mathfrak{M}$-coloring of $\mathbf{T}$, the groups obtained using c and c' are conjugate in $\mathrm{Aut}(\mathbf{T})$ (by any element of $\mathrm{Aut}(\mathbf{T})$ changing c into c', which always exists). For this reason, and since c is fixed, we will suppress it from the notations. In fact, from now on, Γ will denote $\mathrm{Aut}(\mathfrak{M})$ and G will denote $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$.

This construction is the infinite-dimensional analogue of the one carried out in [BM00, Sec. 3]. Groups arising from this construction are known as *Burger-Mozes universal groups*, at least in the finite-dimensional case, since the seminal paper [BM00]. The terminology is motivated by the following (adapted from the original result [BM00, Prop. 3.2.2.] in the finite dimensional case):

Proposition 1.3. *Suppose $\mathfrak{M}$ is transitive and let $H \leqslant \mathrm{Aut}(T)$ be a vertex-transitive subgroup such that for every $x \in \mathbf{T}$, the action $H \curvearrowright E_{\mathbf{T}}(x)$ is isomorphic to $\Gamma \curvearrowright \mathfrak{M}$. Then H is contained in a conjugate of G by an element of* $\mathrm{Aut}(\mathbf{T})$.

Interestingly, this construction predates [BM00] by at least two decades. Indeed, it appears in [Ol'80] where Ol'šanskiĭ classifies the irreducible continuous unitary representations of these groups in the finite dimensional case under some transitivity hypothesis, as well as those of $\mathrm{Aut}(\mathbf{T})$. The purpose of the present note is to get a similar classification for infinite dimensional universal Burger-Mozes groups.

We start the study of these groups with a general method for building elements of G. All one needs to provide is the image of an arbitrary vertex and a suitable collection of local actions:

Lemma 1.4. *Let $x, y \in \mathbf{T}$ and let $(\gamma_z)_{z \in \mathbf{T}} \in \Gamma^{\mathbf{T}}$. The following assertions are equivalent:*

(i) there exists a unique $g \in G$ such that $g(x) = y$ and $g_z = \gamma_z$ for all $z \in \mathbf{T}$,
(ii) for every $z \sim z'$ in $\mathbf{T}$, $\gamma_z(c(z, z')) = \gamma_{z'}(c(z, z'))$.

Moreover, for every subtree $X \subseteq \mathbf{T}$ and every $(\gamma_z)_{z \in X} \in \Gamma^X$ such that (ii) holds on X, there exists $g \in G$ such that $g(x) = y$ and $g_z = \gamma_z$ for all $z \in X$.

Proof. □

Here are the different notions of transitivity we will be working with.

Definition 1.5. $\mathfrak{M}$ is said to be

1. *transitive* if the action $\Gamma \curvearrowright \mathfrak{M}$ has only one orbit.
2. *2-transitive* if for every $a \neq b$, $a' \neq b' \in \mathfrak{M}$, there exists $\gamma \in \Gamma$ such that $\gamma(a) = a'$ and $\gamma(b) = b'$.
3. *primitive* if the action $\Gamma \curvearrowright \mathfrak{M}$ is transitive and for every (equivalently, some) $a \in \mathfrak{M}$, Γ_a is a maximal subgroup of Γ.

Similarly, given $H \leqslant \mathrm{Aut}(\mathbf{T})$, the action of H on $\mathbf{T}$ is said to be:

1. *vertex-transitive* if it has only one orbit.
2. *edge-transitive* if for every $x, y, x', y' \in \mathbf{T}$ with $x \sim y$ and $x' \sim y'$, there exists $h \in H$ such that $h(x) = x'$ and $h(y) = y'$.
3. *2-transitive* if for every $x, y, x', y' \in \mathbf{T}$ with $d(x, y) = d(x', y')$, there exists $h \in H$ such that $h(x) = x'$ and $h(y) = y'$.

We will be going back and forth between the global action $G \curvearrowright \mathbf{T}$ and the local action $\Gamma \curvearrowright \mathfrak{M}$. Given $x \in \mathbf{T}$, a set $A \subseteq E_{\mathbf{T}}(x)$ of edges around x identifies (using the coloring c) with a subset of $\mathfrak{M}$. Next is how we will make identifications in the other direction:

Definition 1.6. Given $a \in \mathfrak{M}$ and $x \in \mathbf{T}$, we will denote by a_x the unique vertex of $\mathbf{T}$ adjacent to x such that the corresponding edge bears the label a, i.e. such that $c(x, a_x) = a$. Similarly, if A is a subset of $\mathfrak{M}^{\mathrm{eq}}$, it identifies *at* x with the following subtree of $\mathbf{T}$:

$$A_x := \{x\} \cup \{y \in V_{\mathbf{T}}(x),\ c(x,y) \in A\}\ .$$

Lemma 1.7.

1. *$G \curvearrowright \mathbf{T}$ is vertex-transitive.*
2. *$G \curvearrowright \mathbf{T}$ is edge-transitive* $\iff$ *$\Gamma \curvearrowright \mathfrak{M}$ is transitive.*
3. *$G \curvearrowright \mathbf{T}$ is 2-transitive* $\iff$ *$\Gamma \curvearrowright \mathfrak{M}$ is 2-transitive.*

Proof.

1. Let $x, y \in \mathbf{T}$. By Lemma 1.4, the following conditions define an element h of G sending x to y :
   - $h(x) = y$,
   - $\forall z \in \mathbf{T},\ h_z = \mathrm{id}_{\mathfrak{M}}$.
2. See the case $n = 1$ in the proof of 3.
3. Suppose $G \curvearrowright \mathbf{T}$ is 2-transitive and let $a \neq b, a' \neq b' \in \mathfrak{M}$. Fix any $x \in \mathbf{T}$. Then:

$$d(a_x, b_x) = 2 = d(a'_x, b'_x)\ ,$$

hence there exists $g \in G$ such that $g(a_x) = a'_x$ and $g(b_x) = b'_x$. Necessarily, $g(x) = x$ and g_x is an element of Γ that sends a to a' and b to b'.

Conversely, assume $\Gamma \curvearrowright \mathfrak{M}$ is 2-transitive. Let $x, y, x', y' \in \mathbf{T}$ be vertices satisfying $d(x,y) = d(x',y')$. We reason by induction on $n = d(x,y)$.

- $\underline{n = 1}$:
  Let $a = c(x,y)$ and $a' = c(x',y')$. Since $\Gamma \curvearrowright \mathfrak{M}$ is transitive, there exists $\gamma \in \Gamma$ such that $\gamma(a) = a'$. We can send x to x' using an automorphism g of $\mathbf{T}$ with constant local action γ, i.e. such that $g_z = \gamma$ for all $z \in \mathbf{T}$. Then g belongs to G by Lemma 1.4 and also satisfies $g(y) = y'$.
- $\underline{n \geqslant 1}$:
  Assume that the property is true for distances up to n and that $d(x,y) = n+1$. Let z (resp. z') be the unique element of $[x,y]$ (resp. $[x',y']$) satisfying $d(z,y) = 1$ (resp. $d(z',y') = 1$). Then $d(x,z) = n = d(x',z')$ and, by the induction hypothesis, there is $g \in G$ such that $g(x) = x'$ and $g(z) = z'$.
  Suppose $g(y) \neq y'$, otherwise there is nothing left to prove. Let w be the unique element of $[x',z']$ such that $d(w,z') = 1$. Let $a = c(z', g(y))$, $a' = c(z',y')$ and $b = c(w,z)$, represented on Figure 2.

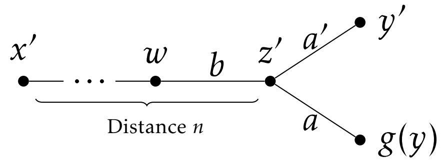

Figure 2. Situation after simplifications.

Then a, a' and b are pairwise distinct elements of $\mathfrak{M}$ hence there exists $\gamma \in \Gamma$ such that $\gamma(a) = a'$ and $\gamma(b) = b$. By Lemma 1.4, the following conditions define an element h of G

- $h(z') = z'$,
- $\forall v \in \mathrm{Cone}_{z'}(w),\ h_v = \mathrm{id}$

– $\forall v \in \mathbf{T}\setminus \mathrm{Cone}_{z'}(w)$, $h_v = \gamma$.

By construction, h fixes $[x', z'] = g([x, z])$ pointwise and sends $g(y)$ to y'. Thus, the element hg of G satisfies:

$$hg(x) = h(x') = x' \qquad \text{and} \qquad hg(y) = y'$$

and the proof is finished.

□

As in the finite-dimensional case (see [BM00, Sec. 3.2]), it is possible to give a precise structural description of vertex stabilizers in G. Indeed, assume $\mathfrak{M}$ is transitive and fix arbitrary reference points $a \in \mathfrak{M}$ and $x \in \mathbf{T}$. For every $n \geqslant 1$, let $G_n \leqslant \mathrm{Sym}(S_{\mathbf{T}}(x, n))$ be the permutation group induced by the action of G_x on $S_{\mathbf{T}}(x, n)$. Then:

1. $(G_n)_{n\geqslant 1}$ is naturally an inverse sequence and G is the inverse limit of $(G_n)_{n\geqslant 1}$,
2. $G_1 \simeq \Gamma$ and for every $n \geqslant 1$, $G_{n+1} \simeq G_n \ltimes \Gamma_a^{S(x,n)}$ (as permutation groups).

Recall that a subset E of a topological group H is *Roelcke-precompact* if for every open neighborhood U of 1 in H, there exists a finite subset $F \subseteq H$ such that $E \subseteq UFU$. A group is *Roelcke-precompact* if it is Roelcke-precompact as a subset of itself and it is *locally Roelcke-precompact* if it admits a Roelcke-precompact non-empty open subset. Since direct-products, semi-direct products and inverse limits of Roelcke-precompact groups are Roelcke-precompact [Tsa12, Prop. 2.2], the previous structural claims imply the following:

Proposition 1.8. *Assume $\mathfrak{M}$ is transitive and such that* $\mathrm{Aut}(\mathfrak{M})$ *is Roelcke-precompact. Then* $G := \mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ *is locally Roelcke-precompact. More precisely, the stabilizers of vertices and edges in G are Roelcke-precompact.*

By the classical Ryll-Nardzewski theorem, this applies in particular when $\mathfrak{M}$ is $\aleph_0$-categorical.

Corollary 1.9. *Assume $\mathfrak{M}$ is 2-transitive and such that* $\mathrm{Aut}(\mathfrak{M})$ *is Roelcke-precompact and let $x \in \mathbf{T}$. Given $K \subset G$ (resp. G^+), the following properties are equivalent:*

(i) K is coarsely bounded in G (resp. G^+),
(ii) K is Roelcke-precompact in G (resp. G^+),
(iii) $K \cdot x$ is bounded in $\mathbf{T}$.

Proof. Since G and G^+ are locally Roelcke-precompact by Proposition 1.8, we refer to [Zie21, Th. 14] for the equivalence between (i) and (ii). We now prove that (ii) and (iii) are equivalent. The proofs for G and G^+ are the same since $G_A = G_A^+$ as soon as $A \subseteq \mathbf{T}$ is non-empty, hence we only prove it for G.

Assume K is Roelcke-precompact in G. Then, G_x being open, there exists finite subset F of K such that $K \subseteq G_x F G_x$. Let $M := \max_{f\in F} d(fx, x)$. If $k \in K$, there exists $u, v \in G_x$ and $f \in F$ such that $k = ufv$. Then $d(kx, x) = d(fvx, u^{-1}x) = d(fx, x) \leqslant M$.

Regarding the converse, observe the following: For every $g \in G$, the double coset $G_x g G_x$ is determined by $d(g(x), x)$. Indeed, suppose $g, h \in G$ satisfy $d(g(x), x) = d(h(x), x)$. Then, since $G \curvearrowright \mathbf{T}$ is 2-transitive by Item 3 of Lemma 1.7, there exists $u \in G$ such that $ug(x) = h(x)$ and $u(x) = x$. This exactly means $h \in G_x g G_x$, i.e. $G_x g G_x = G_x h G_x$.

Now since $K \cdot x$ is bounded, the set $\{d(kx, x),\ k \in K\}$ is finite. Fix $F \subseteq K$ finite covering all the possibilities. By what precedes, $K \subseteq G_x F G_x$. Now G is locally Roelcke-precompact by Proposition 1.8 hence products of Roelcke-precompact subsets of G are Roelcke-precompact [Zie21, Prop. 12]. Since G_x is Roelcke-precompact in G by Proposition 1.8 and since finite sets are always Roelcke-precompact, so is K. □

1.2. **Hilbert spaces and unitary representations.** We recall here some basic results about Hilbert spaces and unitary representations of permutation groups. First, the celebrated Alaoglu-Birkhoff theorem will help us locate invariant vectors.

Theorem 1.10. *Let $\mathcal{H}$ be a Hilbert and let K be any subgroup of $\mathcal{U}(\mathcal{H})$. Let p be the orthogonal projection on $\mathcal{H}^K$, the closed subspace of K-invariant vector. Then, for every $\eta \in \mathcal{H}$, $p\eta$ belongs to* *the norm-closed convex hull* $\overline{\mathrm{Conv}}(K\cdot\eta)$ *of* $K\cdot\eta$*. More precisely,* $p\eta$ *is the only* K*-invariant vector in* $\overline{\mathrm{Conv}}(K\cdot\eta)$.

Next is a convergence theorem for projections associated with a directed collection of subspaces of a Hilbert space, which will allow us to carry out approximation arguments.

Lemma 1.11. *Let* $\mathcal{H}$ *be a Hilbert space,* $(I,\leqslant)$ *a directed set and* $(\mathcal{K}_i)_{i\in I}$ *a directed union of closed subspaces of* $\mathcal{H}$*. For every* $i\in I$*, denote by* p_i *the orthogonal projection onto* $\mathcal{K}_i$*. Let* $\mathcal{K}:=\overline{\bigcup_{i\in I}\mathcal{K}_i}$ *and let* p *be the orthogonal projection onto* $\mathcal{K}$*. Then* (p_i) *converges to* p *in the strong operator topology, i.e.:*

$$\forall\xi\in\mathcal{H},\ \|p_i\xi-p\xi\|\underset{i}{\longrightarrow}0\ .$$

We now recall classical definitions about permutation groups and their representations. Let Ω be a countable set and let $\mathrm{Sym}(\Omega)$ be the group of all bijections of Ω. In this paper, Ω will be either $\mathbf{T}$, $\mathbf{T}^{\mathrm{eq}}$, $\mathfrak{M}$ or $\mathfrak{M}^{\mathrm{eq}}$. We equip $\mathrm{Sym}(\Omega)$ with the *permutation group topology*, i.e. the Polish group topology given by pointwise convergence on Ω discrete. Let K be a *closed permutation group* over Ω, i.e. a closed subgroup of $\mathrm{Sym}(\Omega)$, and let $A\subseteq\Omega$. We will write K_A for the *pointwise stabilizer* of A in K and $\widetilde{K}_A$ for the *setwise stabilizer* of A in K.

A *continuous unitary representation* of K is a continuous linear isometric action of K on a Hilbert space. It is *non-zero* if $\mathcal{H}\neq 0$. Throughout this paper, "representation" will always mean "continuous unitary representation". Let $\pi\colon K\curvearrowright\mathcal{H}$ and $\sigma\colon K\curvearrowright\mathcal{K}$ be two representations of K. An *isomorphism* between π and σ is a K-equivariant surjective isometry $\mathcal{H}\longrightarrow\mathcal{K}$. We will say that σ is (isomorphic to) a subrepresentation of π, and write $\sigma\subseteq\pi$, if there exist a K-equivariant isometric embedding $\mathcal{K}\longrightarrow\mathcal{H}$. The subrepresentation σ is proper if $\mathcal{K}$ identifies as a proper subspace of $\mathcal{K}$ via the embedding. The representation π is *irreducible* if $\mathcal{H}\neq 0$ and π admits no non-zero proper subrepresentation. The *unitary dual* of K is the set of isomorphism classes of irreducible representations of K.

If A is finite, we will write $\mathcal{H}_A$ for the closed subspace of $\mathcal{H}$ consisting of K_A-invariant vectors. If A is infinite, we set:

$$\mathcal{H}_A=\overline{\bigcup_{B\subseteq A\text{ finite}}\mathcal{H}_B}\ .$$

If $A=\{a\}$ is a singleton, we will write $\mathcal{H}_a$ instead. Finally, p_A (or p_a) will denote the orthogonal projection on $\mathcal{H}_A$. When several representations are at play, we will specify them in the notation and write p_A^π instead.

Lemma 1.12. *Let* K *be a closed permutation group over a set* Ω *and let* $\pi\colon K\curvearrowright\mathcal{H}$*. Then* $\mathcal{H}=\mathcal{H}_\Omega$*, i.e.:*

$$\bigcup_{A\subseteq\Omega\text{ finite}}\mathcal{H}_A\quad\text{is dense in }\mathcal{H}\ .$$

The following lemma is a well known fact about induced representations (for the definition and basic properties of induction, we refer to [Fol95, Chap. 6]). Since we will only be inducing from *open* subgroups $U\leqslant G$, we will always use the *counting measure* on the discrete countable space G/U.

Lemma 1.13. *Let* K *be a topological group and let* $U\leqslant K$ *be an open subgroup. Let* $\sigma\colon U\curvearrowright\mathcal{K}$ *be a unitary representation of* U *on a Hilbert space* $\mathcal{K}$*. Consider the representation* $\pi:=\mathrm{Ind}_U^K(\sigma)$ *of* K *and denote by* $\mathcal{H}$ *the underlying Hilbert space. Let*

$$\mathcal{H}_0=\{f\in\mathcal{H},\ \mathrm{Supp}\,f\subseteq U\}.$$

Then $\mathcal{H}_0$ *is stable under the action of* U *and* $\pi\big|_U\colon U\curvearrowright\mathcal{H}_0$ *is isomorphic to* σ*.*

Definition 1.14. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$ be closed subspaces of $\mathcal{H}$ such that $\mathcal{H}_3\subseteq\mathcal{H}_1\cap\mathcal{H}_2$. We will write $\mathcal{H}_1\ominus\mathcal{H}_3$ for the orthogonal complement of $\mathcal{H}_3$ in $\mathcal{H}_1$ and:

$$\mathcal{H}_1\perp_{\mathcal{H}_3}\mathcal{H}_2\quad\text{if}\quad\mathcal{H}_1\ominus\mathcal{H}_3\perp\mathcal{H}_2\ominus\mathcal{H}_3.$$

Denoting by p_1,p_2,p_3 the orthogonal projections on $\mathcal{H}_1,\mathcal{H}_2$ and $\mathcal{H}_3$ respectively, this is equivalent to $p_1p_2=p_3$

2. Irreducible representations of $\mathrm{Aut}_{\mathfrak{M}}(T)$

The study of the irreducible continuous unitary representations of G splits into two very distinct parts. In the classical terminology, an irreducible representations $\pi\colon G \curvearrowright \mathcal{H}$ is said to be *spherical* if there exists $x \in \mathbf{T}$ such that $\mathcal{H}_x \neq 0$. Ol'šanskiĭ's classification of spherical irreducible representation of $\mathrm{Aut}(\mathbf{T})$, which corresponds to $\mathfrak{M}$ being the trivial structure, actually extends with the same proof to every 2-transitive subgroup of $\mathrm{Aut}(\mathbf{T})$. In particular, it applies to G provided that $\mathfrak{M}$ is 2-transitive (see Lemma 1.7). Moreover, in this, case, this part of the unitary dual of G is actually independent of $\mathfrak{M}$:

Theorem 2.1 ([Ol'80])**.** *Let $H \leqslant \mathrm{Aut}(\mathbf{T})$ be a 2-transitive subgroup (e.g. $H = G$ assuming $\mathfrak{M}$ is 2-transitive). Let π be a spherical irreducible representation of $\mathrm{Aut}(\mathbf{T})$. Then $\pi\big|_H$ is a spherical irreducible representation of H. Moreover, every spherical irreducible representation of H arises uniquely in this manner.*

In other words, the correspondence $\pi \longmapsto \pi\big|_H$ that sends a representation of $\mathrm{Aut}(\mathbf{T})$ to its restriction to H induces a bijective correspondence between the spherical irreducible representations of $\mathrm{Aut}(T)$ and H.

Proof. This follows easily from [Ol'80, Sec. 5.3–5.6], the proof of which applies word for word to $H \leqslant \mathrm{Aut}(\mathbf{T})$ under the 2-transitivity hypothesis. $\square$

On the other hand, the *non-spherical* irreducible representations of G heavily depend on $\mathfrak{M}$. In particular, if $\sigma : \Gamma \curvearrowright \mathcal{H}$ is a unitary representation of Γ, then it can be pulled to a representation of G_x via the restriction morphism $G_x \longrightarrow \Gamma$ given by $g \longmapsto g_x$. The restriction morphism being surjective, this representation of G_x retains all the richness of σ and can in turn be induced to G. For that reason, the representation theory of G is at least as complicated as that of Γ hence we must restrict our attention to instances of the construction where the representation theory of Γ is understood. A good candidate for such a class are the *Roelcke-precompact* closed permutation groups. Indeed, the unitary representations of these groups have been fully classified in [Tsa12]. Since we will need transitivity hypothesis, we will actually work with Γ being presented as the automorphism group of an $\aleph_0$-categorical structure. Recall that, by the Ryll-Nardzewski Theorem, for $\mathfrak{M}$ countable and transitive, Γ is Roelcke-precompact if and only if $\mathfrak{M}$ is $\aleph_0$-categorical.

The notion of *algebraicity*, appearing at different levels, will be key. Here is the permutation group theoretic definition of algebraicity. Fortunately, when $\mathfrak{M}$ is $\aleph_0$-categorical, it coincides with the model theoretical definition, both for $\Gamma \curvearrowright \mathfrak{M}$ and $\Gamma \curvearrowright \mathfrak{M}^{\mathrm{eq}}$.

Definition 2.2. Let H be a closed permutation group over a set Ω and let A be a subset of Ω.

1. An element $x \in \Omega$ is said to be *algebraic* (resp. *definable*) *over A* (with respect to the action of H on Ω) if there exists a finite subset $B \subseteq A$ such that the orbit $H_B \cdot x$ if finite (resp x is fixed by H_B). The *algebraic closure* (resp. *definable closure*) of A, denoted $\mathrm{acl}_\Omega(A)$ (resp. $\mathrm{dcl}_\Omega(A)$), is the set of elements of Ω that are algebraic (resp. definable) over A. The set A is *algebraically closed* if it is equal to its algebraic closure.
2. the set A is said to be *definable* (with respect to the action of H on Ω) if $\widetilde{H}_D$ is open in H.
3. A is said to be *locally H-finite* if its intersection with every orbit of the action of H on Ω is finite.

In the special case of the action $\Gamma \curvearrowright \mathfrak{M}^{\mathrm{eq}}$, we will write $\mathcal{D}_{\mathcal{M}}$ for the set of definable, locally finite and algebraically subsets of $\mathfrak{M}^{\mathrm{eq}}$. When $\mathfrak{M}$ is $\aleph_0$-categorical, understanding $\mathcal{D}_{\mathcal{M}}$ is key for understanding the representation theory of Γ. Indeed, the algebraically closed subset of $\mathfrak{M}^{\mathrm{eq}}$ one needs to input in Lemma 2.4 in order to recover a classification theorem such as in [Tsa12] are exactly the elements of $\mathcal{D}_{\mathcal{M}}$. They form a lattice of sets $(\mathcal{D}_{\mathcal{M}}, \subseteq, \wedge, \vee)$ where $A \wedge B = A \cap B$ and $A \vee B = \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A \cup B)$ for every $A, B \in \mathfrak{M}^{\mathrm{eq}}$. The following nice properties of this lattice are extracted from Section 2.1 of the preprint [Tsa24].

Proposition 2.3. *Assuming $\mathfrak{M}$ is $\aleph_0$-categorical, the following properties hold:*

1. *The elements of $\mathcal{D}_{\mathcal{M}}$ are exactly the subsets of $\mathfrak{M}^{\mathrm{eq}}$ of the form $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(e)$ for some $e \in \mathfrak{M}^{\mathrm{eq}}$ (equivalently, of the form $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A)$ for some finite $A \subseteq \mathfrak{M}^{\mathrm{eq}}$).*

2. *For every $D \in \mathcal{D}_{\mathcal{M}}$, the trace $D \cap \mathfrak{M}$ is finite.*
3. *$(\mathcal{D}_{\mathcal{M}}, \subseteq, \wedge, \vee)$ is a well-founded ordered lattice, i.e. every non-empty subset of $\mathcal{D}_{\mathcal{M}}$ admits a minimal element.*

From the classification of [Tsa12], the authors of [JT22] extracted the following lemma (Prop. 3.4 therein), which states that $\aleph_0$-categorical structures are *dissociated* in a slightly more general sense than that of [BJJ24]. This fundamental property, tightly tying the permutation group structure of Γ to its representation theory, is enough to recover the full classification theorem in a straightforward way (see the proof of Corollary 2.21 below).

Lemma 2.4. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical. Let $\pi\colon \Gamma \curvearrowright \mathcal{H}$ be a unitary representation of Γ. For all algebraically closed subsets A, B of $\mathfrak{M}^{\mathrm{eq}}$,*

$$\mathcal{H}_A \perp_{\mathcal{H}_{A\cap B}} \mathcal{H}_B \,,$$

i.e. $p_A p_B = p_{A\cap B}$.

As one can see from the above lemma, understanding the representation theory of Γ when $\mathfrak{M}$ is $\aleph_0$-categorical involves the *imaginaries* of $\mathfrak{M}$, i.e. equivalence classes of definable equivalence relations on $\mathfrak{M}$. See [Hod93, Sec. 4.3] for the definition of imaginaries and the structure $\mathfrak{M}^{\mathrm{eq}}$ associated to $\mathfrak{M}$.

To take them into account in the present construction, we need to introduce a slightly larger colored tree than $\mathbf{T}$. In this way, let $\mathbf{T}^{\mathrm{eq}}$ be the tree obtained from $\mathbf{T}$ by adding a countably infinite collection of *leaves* around each of its vertices. Note that $\mathbf{T} \subseteq \mathbf{T}^{\mathrm{eq}}$. We also extend c to an $\mathfrak{M}^{\mathrm{eq}}$–coloring of $\mathbf{T}^{\mathrm{eq}}$ in such a way (essentially unique) that:

$$\forall x \in \mathbf{T},\ [c\colon E_{\mathbf{T}^{\mathrm{eq}}}(x) \longrightarrow \mathfrak{M}^{\mathrm{eq}} \text{ is a bijection}].$$

The resulting colored tree will still be denoted $\mathbf{T}^{\mathrm{eq}}$ and is represented in Figure 1 (right). We extend definition 1.6 to that context.

Convention. In graphical representations such as in Figure 1, solid vertices and thick edges, in black, will always correspond to elements of $\mathbf{T}$ or labels in $\mathfrak{M}$. On the other hand, hollowed vertices and thin edges, in blue, will always correspond to elements of $\mathbf{T}^{\mathrm{eq}} \backslash \mathbf{T}$ or labels in $\mathfrak{M}^{\mathrm{eq}} \backslash \mathfrak{M}$.

We will adopt similar conventions in writing: x, y, z will denote vertices of $\mathbf{T}$ or $\mathbf{T}^{\mathrm{eq}}$ and a, b, c will denote elements of $\mathfrak{M}$ or $\mathfrak{M}^{\mathrm{eq}}$. Finally, X, Y, Z will denote subtrees of $\mathbf{T}$ or $\mathbf{T}^{\mathrm{eq}}$ while A, B, C will denote subsets of $\mathfrak{M}$ or $\mathfrak{M}^{\mathrm{eq}}$.

Remark 2.5. It is a classical fact that the action $\mathrm{Aut}(\mathfrak{M}) \curvearrowright \mathfrak{M}$ naturally extends to an action on $\mathfrak{M}^{\mathrm{eq}}$, yielding a canonical identification $\mathrm{Aut}(\mathfrak{M}) \simeq \mathrm{Aut}(\mathfrak{M}^{\mathrm{eq}})$ (see [Hod93, Th. 4.3.3]). For that reason, the action $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T}) \curvearrowright \mathbf{T}$ naturally extends to an action on $\mathbf{T}^{\mathrm{eq}}$.

Next is an explicit link between the actions $G \curvearrowright \mathbf{T}^{\mathrm{eq}}$ and $\Gamma \curvearrowright \mathfrak{M}^{\mathrm{eq}}$.

Lemma 2.6. *Let X be a subtree of $\mathbf{T}^{\mathrm{eq}}$ and $x \in X$. Let $Z = X \cup V_{\mathbf{T}^{\mathrm{eq}}}(x)$ and let $A = E_X(x)$ be seen as a subset of $\mathfrak{M}^{\mathrm{eq}}$. The restriction map*

$$G_X \longrightarrow \Gamma_A \,, \qquad g \longmapsto g_x$$

is continuous, open and surjective. In particular, $G_X/G_Z \simeq \Gamma_A$ (as topological groups).

Proof. We first show surjectivity. Let $\gamma \in \Gamma_{V_X(x)}$. We will define an extension $g \in G$ of γ. Reasoning by induction, we assume g is defined on $B_{\mathbf{T}}(x, n)$ for some $n \geqslant 1$ and extend it to $B_{\mathbf{T}}(x, n+1)$. To that aim, we fix $y \in S_{\mathbf{T}}(x, n)$ and extend g to $V_{\mathbf{T}}(y)$. It suffices to choose a local action at y, i.e. specify g_y. If $y \in X$, then it is fixed by g and we set $g_y = \mathrm{id}$. Otherwise, let z be the unique neighbor of y with $d(z, x) = n - 1$. We necessarily have $z \in \mathbf{T}$. We propagate the local action of g at z, i.e. we set $g_y = g_z$.

Openness and continuity are clear and imply the last isomorphism. □

Corollary 2.7. *Let X be a subtree of $\mathbf{T}^{\mathrm{eq}}$ and $x \in X$. Write $A = E_X(x)$, seen as a subset of $\mathfrak{M}^{\mathrm{eq}}$. For every $y \in V_{\mathbf{T}^{\mathrm{eq}}}(x)$, the map*

$$G_X \cdot y \longrightarrow \Gamma_A \cdot c(x, y) \,, \qquad g(y) \longmapsto g_x(c(x, y))$$

is a bijection.

The following lemma clarifies what we will mean when we ask of a representation to be trivial on a stabilizer.

Lemma 2.8. *Let X be a subtree of $\mathbf{T}^{\mathrm{eq}}$. For every continuous unitary representation $\sigma\colon \widetilde{G}_X \longrightarrow \mathcal{U}(\mathcal{H})$, the following properties are equivalent:*

(i) $G_X \subseteq \ker\sigma$,
(ii) $\mathcal{H} = \mathcal{H}_X$.

Proof. Assume $G_X \subseteq \ker\sigma$. Then σ factors as a continuous unitary representation of $\widetilde{G}_X/G_X$. Moreover, the natural map $\widetilde{G}_X/G_X \longrightarrow \mathrm{Sym}(X)$ is injective, continuous and has closed image. Since the groups are Polish, it is a homeomorphic embedding that identifies $\widetilde{G}_X/G_X$ with a closed permutation group over X. Then $\mathcal{H} = \mathcal{H}_X$ follows from Lemma 1.12. The converse is trivial. □

We now define the class of sets that will play for $G \curvearrowright \mathbf{T}^{\mathrm{eq}}$ the role that $\mathcal{D}_{\mathcal{M}}$ plays for $\Gamma \curvearrowright \mathfrak{M}^{\mathrm{eq}}$.

Definition 2.9. A subtree X of $\mathbf{T}^{\mathrm{eq}}$ is *admissible* if it is definable, locally G-finite and algebraically closed in $G \curvearrowright \mathbf{T}^{\mathrm{eq}}$. An admissible subtree X is *trivial* if $\widetilde{G}_X = G$ or $\widetilde{G}_X = G_x$ for some $x \in \mathbf{T}$. The set of admissible subtrees of $\mathbf{T}^{\mathrm{eq}}$ will be denoted $\mathrm{Adm}(\mathbf{T}^{\mathrm{eq}})$.

Just as elements of $\mathcal{D}_{\mathcal{M}}$ can be used to classify the irreducible representations of $\mathrm{Aut}(\mathfrak{M})$ if $\mathfrak{M}$ is $\aleph_0$-categorical, we will use the admissible subtrees to get a similar classification for G. We begin the study of $\mathrm{Adm}(\mathbf{T}^{\mathrm{eq}})$ with a very elementary observation:

Proposition 2.10. *For every admissible subtree X of $\mathbf{T}^{\mathrm{eq}}$,*

1. *if X is non-empty, $X \cap \mathbf{T} \neq \emptyset$,*
2. *the leaves of X all lie in $\mathbf{T}^{\mathrm{eq}} \backslash \mathbf{T}$.*

Proof.

1. If $X \cap \mathbf{T} = \emptyset$, X is a singleton $\{x\}$. Moreover, $x \in \mathbf{T}^{\mathrm{eq}} \backslash \mathbf{T}$ hence is adjacent to a unique vertex $y \in \mathbf{T}$. Then G_X fixes $y \notin X$, contradicting the fact that X is algebraically closed.
2. Suppose X is non-empty and let $x \in X \cap \mathbf{T}$. Since X is algebraically closed in $\mathbf{T}^{\mathrm{eq}}$, we have $\mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset)_x \subseteq V_X(x)$ and x is not a leaf in X. Indeed, because of the formal definition of $\mathfrak{M}^{\mathrm{eq}}$, $|\mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}| \geqslant 2$.

□

The next step in relating elements of $\mathcal{D}_{\mathcal{M}}$ to admissible subtrees of $\mathbf{T}^{\mathrm{eq}}$ is the following:

Proposition 2.11. *Let X be a subtree of $\mathbf{T}^{\mathrm{eq}}$ such that $X \cap \mathbf{T} \neq \emptyset$. The following equivalences hold:*

1. *X is locally G-finite* $\iff$ *$X \cap \mathbf{T}$ is finite and $E_X(x)$ is locally Γ-finite in $\mathfrak{M}^{\mathrm{eq}}$ for every $x \in X \cap \mathbf{T}$.*
2. *X is algebraically closed* $\iff$ *$E_X(x)$ is algebraically closed in $\mathfrak{M}^{\mathrm{eq}}$ for every $x \in X \cap \mathbf{T}$.*

Moreover, if $X \cap \mathbf{T}$ is finite:

3. *X is definable* $\iff$ *$E_X(x)$ is definable in $\mathfrak{M}^{\mathrm{eq}}$ for every $x \in X \cap \mathbf{T}$.*

Proof.

1. Assume X is locally G-finite. Then $X \cap \mathbf{T}$ is indeed finite since $\mathbf{T}$ is a single orbit under the action of G (Item 1 of Lemma 1.7). Let $x \in X \cap \mathbf{T}$ and let A be the subset of $\mathfrak{M}^{\mathrm{eq}}$ corresponding to $E_X(x)$. Let us show that A is Γ-finite.

   Fix $a \in \mathfrak{M}^{\mathrm{eq}}$. Then $\Gamma \cdot a \simeq G_x \cdot a_x$ by Corollary 2.7, hence:
$$\Gamma \cdot a \cap A \simeq G_x \cdot a_x \cap V_X(x) \subseteq G \cdot a_x \cap X.$$
   Since X is locally G-finite, $G \cdot a_x \cap X$ is finite and so is $\Gamma \cdot a \cap A$.

Conversely, assume that $X \cap \mathbf{T}$ is finite and that $E_X(z)$ is locally Γ-finite for every $z \in X \cap \mathbf{T}$. Let $x \in \mathbf{T}^{\mathrm{eq}}$. Since $G \cdot \mathbf{T} = \mathbf{T}$ and $X \cap \mathbf{T}$ is assumed to be finite, it suffices to deal with the case $x \in \mathbf{T}^{\mathrm{eq}} \backslash \mathbf{T}$. Then x is a leaf, adjacent to a unique vertex $y \in \mathbf{T}$. Setting $a = c(x, y)$, we have:

$$G \cdot x \cap X = \bigcup_{z \in X \cap \mathbf{T}} G \cdot x \cap V_X(z) = \bigcup_{z \in X \cap \mathbf{T}} (\Gamma \cdot a)_z \cap V_X(z),$$

which is finite, as a finite union of finite sets.

2. Assume X is algebraically closed with respect to the action $G \curvearrowright \mathbf{T}^{\mathrm{eq}}$. Fix $x \in X$ and let A be the subset of $\mathfrak{M}^{\mathrm{eq}}$ corresponding to $E_X(x)$. Let us show that A is algebraically closed.

Let $A' \subseteq A$ be finite. For every $a \in \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A')$, Corollary 2.7 yields:

$$\Gamma_{A'} \cdot a \simeq G_{A'_x} \cdot a_x \, .$$

In particular, $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A')_x \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(A'_x) \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) = X$. Thus $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A')_x \subseteq V_X(x) \cup \{x\}$ i.e. $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A') \subseteq A$.

Conversely, assume that $E_X(x)$ is algebraically closed in $\mathfrak{M}^{\mathrm{eq}}$ for every $x \in X \cap \mathbf{T}$. Let X' be a finite subset of X. Up to replacing X' with a larger finite subset of X, we assume it is a subtree of $\mathbf{T}^{\mathrm{eq}}$ and that $X' \cap \mathbf{T} \neq \emptyset$. Let $x \in \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X')$. Let y be the unique vertex of X' witnessing $d(x, y) = d(x, X')$. We reason by induction on $n = d(x, y)$.

- $\underline{n = 0}$:
  Then $y \in X$ and there is nothing to prove.
- $\underline{n > 0}$:
  Let z be the unique vertex of $\mathbf{T}^{\mathrm{eq}}$ adjacent to y that satisfies $d(z, x) = n - 1$. Let A be the subset of $\mathfrak{M}^{\mathrm{eq}}$ corresponding to $E_X(y)$ and A' the one corresponding to $E_{X'}(y)$. By Corollary 2.7,
  $$G_{X'} \cdot z \simeq \Gamma_{A'} \cdot c(y, z)$$
  Hence $c(y, z) \in \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A') \subseteq A$ and $z \subseteq X$. We can thus add z to X' and apply the induction hypothesis to get $x \in X$.

3. Assume that X is definable (the finiteness hypothesis is not needed for this direction). Let $x \in X \cap \mathbf{T}$ and let A be the subset of $\mathfrak{M}^{\mathrm{eq}}$ corresponding to $E_X(x)$. By Lemma 2.6, the image of $\widetilde{G}_X \cap G_x$ under the restriction map $g \longmapsto g_x$ is open. Since, clearly, it is included in $\widetilde{\Gamma}_A$, A is definable in $\mathfrak{M}^{\mathrm{eq}}$.

Conversely, let X be a subtree of $\mathbf{T}^{\mathrm{eq}}$ such that $X \cap \mathbf{T}$ is finite and $E_X(x)$ is definable in $\mathfrak{M}^{\mathrm{eq}}$ for every $x \in X \cap \mathbf{T}$. Since X is a tree, we can assume $X \cap \mathbf{T}$ is non-empty. Note that:

$$\widetilde{G}_X \geqslant \bigcap_{x \in X \cap \mathbf{T}} \widetilde{G}_{V_X(x) \cup \{x\}} \, .$$

The intersection being finite, it suffices to show that for every definable subset $A \subseteq \mathfrak{M}^{\mathrm{eq}}$, $\widetilde{G}_{A_x}$ is open in G. Now if A is a definable subset of $\mathfrak{M}^{\mathrm{eq}}$, then $\widetilde{G}_{A_x}$ is the inverse image of $\widetilde{\Gamma}_A$ under the restriction map $G_x \longrightarrow \Gamma$. This map being continuous by Lemma 2.6, $\widetilde{G}_{A_x}$ is open.

□

As a consequence, we get a local characterization of admissible subtrees.

Corollary 2.12. *Let X be a subtree of $\mathbf{T}^{\mathrm{eq}}$. The following conditions are equivalent:*

(i) X is admissible,

(ii) $X \cap \mathbf{T}$ is finite and for every $x \in X \cap \mathbf{T}$, the subset $E_X(x)$ of $\mathfrak{M}^{\mathrm{eq}}$ is definable, locally Γ-finite and algebraically closed.

Moreover, if X is admissible and non-trivial, then $\widetilde{G}_X$ is Roelcke-precompact.

Proof. The equivalences follow directly from Proposition 2.11.

As for the last statement, it is a classical fact that a group which stabilizes a non-empty finite diameter tree fixes either a vertex or an edge. Since $X \cap \mathbf{T}$ is finite, X has finite diameter and the claim follows from Proposition 1.8 and from the fact that an open subgroup of a Roelcke-precompact group is also a Roelcke-precompact group. □

Similarly to what happens with elements of $\mathcal{D}_{\mathcal{M}}$ in $\mathfrak{M}^{\mathrm{eq}}$, admissible subtrees of $\mathbf{T}^{\mathrm{eq}}$ are always infinite but are algebraic closures of finite subtrees if $\mathfrak{M}$ is $\aleph_0$-categorical.

Lemma 2.13. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical. Let X be a non-trivial admissible subtree of $\mathbf{T}^{\mathrm{eq}}$. There exists a finite subtree $X_0 \subseteq X$ such that $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0) = X$. Moreover, every such X_0 satisfies $G_{X_0} \leqslant \widetilde{G}_X$ with finite index.*

Proof. Using Item 1 in Proposition 2.3, for every $x \in X \cap \mathbf{T}$, there exists $e(x) \in \mathfrak{M}^{\mathrm{eq}}$ such that $E_X(x) \simeq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(e(x))$. Define:

$$X_0 = (X \cap \mathbf{T}) \cup \{e(x)_x,\ x \in X \cap \mathbf{T}\},$$

which is a finite subtree of X. Fix $x \in X \backslash X_0$. We will show that $x \in \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0)$.

Necessarily, x is an element of $\mathbf{T}^{\mathrm{eq}} \backslash \mathbf{T}$ hence a leaf in X. Let y be its unique neighbor, which is an element of $X \cap \mathbf{T}$. Then, using Corollary 2.7:

$$G_{X_0} \cdot x \simeq \Gamma_{e(y)} \cdot c(x,y)\ ,$$

which is finite by choice of $e(y)$. Hence $X \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0)$ and the other inclusion is trivial.

For the finite index assertion, recall that the *commensurator of G_{X_0} in $\widetilde{G}_X$* is given by:

$$\mathrm{Comm}_{\widetilde{G}_X}(G_{X_0}) = \{g \in \widetilde{G}_X,\ \left(G_{X_0} g G_{X_0}\right) \big/ G_{X_0} \text{ and } G_{X_0} \big\backslash \left(G_{X_0} g G_{X_0}\right) \text{ are finite.}\}$$

Lemma 2.7 in [Tsa12] states that open subgroups of a Roelcke-precompact group have finite index in their commensurator. Since $\widetilde{G}_X$ is Roelcke-precompact by Corollary 2.12, proving the the following is enough to conclude:

$$\widetilde{G}_X = \mathrm{Comm}_{\widetilde{G}_X}(G_{X_0}).$$

Let $g \in G_{X_0}$. Since X_0 is finite,

$$\begin{aligned} \left(G_{X_0} g G_{X_0}\right) \big/ G_{X_0} \text{ is finite.} &\iff \forall x \in X_0,\ G_{X_0} g \cdot x \text{ is finite.} \\ &\iff \forall x \in X_0,\ g \cdot x \in \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0) \\ &\iff g \cdot X_0 \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0) = X \\ &\iff g \cdot X \subseteq X \end{aligned}$$

Similarly, $G_{X_0} \big\backslash \left(G_{X_0} g G_{X_0}\right)$ is finite if and only if $g^{-1} \cdot X \subseteq X$ and the claim is proved. ☐

Using the admissible subtrees, we can exhibit a family of irreducible representations of G without any additional hypothesis on $\mathfrak{M}$. To that aim, we first provide a classical argument in the study of invariant spaces associated with induced representations of permutation groups.

Lemma 2.14. *Let X, Y be admissible subtrees of $\mathbf{T}^{\mathrm{eq}}$ and let $\sigma : \widetilde{G}_X \curvearrowright \mathcal{K}$ be a unitary representation of $\widetilde{G}_X$ that is trivial on G_X. Denote $\pi := \mathrm{Ind}_{\widetilde{G}_X}^{G}(\sigma)$ and let $\mathcal{H}$ be the underlying Hilbert space of π. Then:*

$$\mathcal{H}_Y = \{f \in \mathcal{H},\ \forall g \in G,\ [f(g) \neq 0 \Rightarrow gX \subseteq Y]\}.$$

In particular,

$$\mathcal{H}_X = \{f \in \mathcal{H},\ \mathrm{Supp}(f) \subseteq \widetilde{G}_X\}$$

and p_X is the multiplication by the indicator map of $\widetilde{G}_X$.

Proof. Write $\widetilde{\mathcal{H}}_Y := \{f \in \mathcal{H},\ \forall g \in G,\ [f(g) \neq 0 \Rightarrow gX \subset Y]\}$. Since this is a closed subspace of $\mathcal{H}$, it suffices to show that $\mathcal{H}_{Y'} \subseteq \widetilde{\mathcal{H}}_Y$ for every finite subtree $Y' \subseteq Y$ to get $\mathcal{H}_Y \subseteq \widetilde{\mathcal{H}}_Y$. Hence, let $Y' \subseteq Y$ be finite.

Let $f \in \mathcal{H}_{Y'}$ and assume $g \in G$ is such that $f(g) \neq 0$. Then $\|f(\cdot)\|$ is constant on $G_{Y'} g \widetilde{G}_X$. Since $f \in \ell^2(G/\widetilde{G}_X)$, it implies that $\left(G_{Y'} g \widetilde{G}_X\right) \big/ \widetilde{G}_X$ is finite. Let $X_0 \subseteq X$ be as in Lemma 2.13. In particular, $G_{X_0} \leqslant \widetilde{G}_X$ with finite index, hence $\left(G_{Y'} g G_{X_0}\right) \big/ G_{X_0}$ is also finite. Equivalently, since

X_0 is finite, every element of $g(X_0)$ has a finite orbit under the action of $G_{Y'}$. In other words, $g(X_0) \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y') \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y)$. Then:

$$g(X) = g(\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0)) = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(g(X_0)) \subseteq \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y) = Y. \tag{1}$$

Conversely, let $f \in \widetilde{\mathcal{H}}_Y$.

Claim. *We can assume that* $\mathrm{Supp}\, f$ *is a finite union of* $\widetilde{G}_X$ *cosets and that there exists a finite* $\widetilde{G}_X$*-invariant subset* $X' \subseteq X$ *such that:*

$$\forall g \in G,\ f(g) \in \mathcal{H}_{X'}\ . \tag{2}$$

Proof of the claim. Let $\varepsilon > 0$. Since $\mathcal{H}_Y$ is closed, we can assume that $\mathrm{Supp}(f)$ is a finite union of $\widetilde{G}_X$-cosets and pick $h_1, \dots, h_m \in G$ such that $\mathrm{Supp}\, f = \bigsqcup_{1 \leqslant j \leqslant m} h_j \widetilde{G}_X$. By Lemma 2.8, $\mathcal{K} = \mathcal{K}_X$ and, for every $j \leqslant m$, there exists $X_j \subseteq X$ finite and $\xi_j \in \mathcal{K}_{X_j}$ such that $\|\xi_j - f(h_j)\| \leqslant \frac{\varepsilon}{\sqrt{m}}$. Since X is locally G-finite, every element of X has a finite orbit under the action of $\widetilde{G}_X$. In particular,

$$X' \coloneqq \widetilde{G}_X \cdot \bigcup_{1 \leqslant i \leqslant m} X_i$$

is finite and stable under the action of $\widetilde{G}_X$.

Define f_ε as follows:

$$\forall g \in G,\ f_\varepsilon(g) = \begin{cases} \sigma(u^{-1})\xi_j & \text{if } g = h_j u \text{ for some } j \leqslant m \text{ and some } u \in \widetilde{G}_X, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_\varepsilon \in \widetilde{H}_Y$ and $\|f - f_\varepsilon\| \leqslant \varepsilon$. Since $\mathcal{H}_Y$ is closed, it suffices to show that $f_\epsilon(g) \in \mathcal{H}_{X'}$ for every $g \in \mathrm{Supp}\, f_\varepsilon$ to finish the proof of the claim. Let $g \in \mathrm{Supp}\, f_\varepsilon$. There exists $j \leqslant m$ and $u \in \widetilde{G}_X$ such that $g = h_j u$. Then:

$$f_\varepsilon(g) = \sigma(u^{-1})\xi_j \in \mathcal{K}_{u^{-1}(X')} = \mathcal{K}_{X'}.$$

$\square_{\text{claim}}$

Next, let $X_0 \subseteq X$ be as in Lemma 2.13. Since G_{X_0} has finite index in $\widetilde{G}_X$, $\mathrm{Supp}\, f$ is a finite union of G_{X_0}-cosets and we can write:

$$\mathrm{Supp}(f) = \bigcup_{1 \leqslant i \leqslant n} g_i G_{X_0}$$

where $g_1, \dots, g_n \in G$.

Let $Y_0 \subseteq Y$ be as in Lemma 2.13 and define:

$$Y' = Y_0 \cup \bigcup_{1 \leqslant i \leqslant n} g_i(X')$$

which is a finite subset of Y . Indeed, recall that $g_1, \dots, g_n$ belong to $\mathrm{Supp}(f)$ hence satisfy $g_i(X) \subseteq Y$ for every $i \leqslant n$. Moreover, since X' is $\widetilde{G}_X$-stable:

$$\forall i \leqslant n, \forall g \in g_i G_{X_0},\ g(X') = g_i(X') \subseteq Y'. \tag{3}$$

Let us show that f is $G_{Y'}$-invariant. Fix $u \in G_{Y'}$ and $g \in G$.

- <u>If $g \notin g_1 G_{X_0} \cup \dots \cup g_n G_{X_0}$:</u>

  Then $u^{-1}g \notin u^{-1}g_1 G_{X_0} \cup \dots \cup u^{-1} g_n G_{X_0}$. But for every $i \leqslant n$, $g_i(X_0) \subseteq Y'$ hence $u^{-1} g_i G_{X_0} = g_i G_{X_0}$ and $g, u^{-1}g \notin \mathrm{Supp}\, f$. In other words:

  $$(u \cdot f)(g) = f(u^{-1}g) = 0 = f(g).$$

- <u>If $g \in g_i G_{X_0}$ for some $i \leqslant n$:</u>

  Then $u^{-1}g \in G_{Y'}g$. But $G_{Y'}g = gG_{g^{-1}(Y')} \subseteq gG_{X'}$ by (3) and there is $v \in G_{X'}$ such that $u^{-1}g = gv$. Thus:

  $$(u \cdot f)(g) = f(u^{-1}g) = f(gv) = \sigma(v^{-1})(f(g)) = f(g)$$

  where the last equality follows from (2).

For the special case $Y = X$, note that since X is locally G-finite, we have for every $g \in G$:

$$gX \subseteq X \qquad \Longleftrightarrow \qquad gX = X\ .$$

$\square$

The previous lemma can be used to give a list a of irreducible representations of G while also identifying the repetitions in that list.

Recall that if H, H' are conjugate subgroups of G, say $gHg^{-1} = H'$ for some $g \in G$, and if τ is a representation of H' on a Hilbert space $\mathcal{H}$, then $\tau^g : H \curvearrowright \mathcal{H}$ is given by $\tau^g(h) = \tau(ghg^{-1})$ for every $h \in H$.

Proposition 2.15. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical. Let X be an admissible subtree of $\mathbf{T}^{\mathrm{eq}}$ and σ be an irreducible representation of $\widetilde{G}_X$ which is trivial on G_X. Then $\pi := \mathrm{Ind}_{\widetilde{G}_X}^{G}(\sigma)$ is irreducible.*

Moreover, if Y is another admissible subtree of $\mathbf{T}^{\mathrm{eq}}$ and τ is an irreducible representation of $\widetilde{G}_Y$ which is trivial on G_Y and such that $\pi' := \mathrm{Ind}_{\widetilde{G}_Y}^{G}(\tau)$ is isomorphic to π, there exists $g \in G$ such that $g(X) = Y$ and $\sigma \simeq \tau^g$.

Proof. We first prove the irreducibility of π by showing that every non-zero vector in $\mathcal{H}$ is G-cyclic. Indeed, let $f \in \mathcal{H}\backslash\{0\}$. Up to translation, we can assume that $f(1) \neq 0$, or equivalently by Lemma 2.14 that $p_X(f) \neq 0$. By Lemma 1.11 and the Alaoglu-Birkhoff Theorem 1.10, $p_X(f)$ lies in the G-cyclic hull of f hence the following claim finishes the proof of the irreducibility of π:

Claim. *Every non-zero vector of $\mathcal{H}_X$ is G-cyclic in $\mathcal{H}$.*

Proof of the claim. Using Lemmas 2.14 and 1.13, the representation

$$\pi\big|_{\widetilde{G}_X} : \widetilde{G}_X \curvearrowright \mathcal{H}_X$$

is isomorphic to σ hence irreducible. Thus, every non-zero vector of $\mathcal{H}_X$ is $\widetilde{G}_X$-cyclic in $\mathcal{H}_X$, a space which itself is G-cyclic in $\mathcal{H}$. $\square_{\text{claim}}$

Next, assume π and π' are isomorphic and denote by $\mathcal{H}$ and $\mathcal{H}'$ their respective underlying Hilbert space. Then $\mathcal{H}'_X \simeq \mathcal{H}_X \neq 0$. Let $f \in \mathcal{H}'_X\backslash\{0\}$ and $g \in G$ such that $f(g) \neq 0$. By Lemma 2.14, g satisfies $g(Y) \subseteq X$. By symmetry, we can find $h \in G$ such that $h(X) \subseteq Y$. Since X and Y are locally G-finite, this implies $h(X) = Y$. In particular, $h\widetilde{G}_X h^{-1} = \widetilde{G}_Y$. Finally, using Lemmas 2.14 and 1.13 again for the first and last isomorphisms:

$$\sigma \simeq \left(\widetilde{G}_X \overset{\pi}{\curvearrowright} \mathcal{H}_X\right) \simeq \left(\widetilde{G}_X \overset{\pi'}{\curvearrowright} \mathcal{H}'_X\right) \simeq \left(\widetilde{G}_X \overset{\pi'^h}{\curvearrowright} \mathcal{H}'_Y\right) = \left(\widetilde{G}_Y \overset{\pi'}{\curvearrowright} \mathcal{H}'_Y\right)^h \simeq \tau^h\ .$$

$\square$

The previous results does give us a list of irreducible representations of G but it might actually be empty: we have no result regarding the existence of admissible trees yet. We thus investigate the converse of Lemma 2.13. If there is $A \subset \mathfrak{M}$ such that $\mathrm{acl}_{\mathfrak{M}}(A)$ is infinite, or if there is $a, b \in \mathfrak{M}$ distinct such that $\mathrm{acl}_{\mathfrak{M}}(a) = \mathrm{acl}_{\mathfrak{M}}(b)$, then the converse fails for lack of local G-finiteness. However, these are the only obstruction. In particular, if $\mathfrak{M}$ is $\aleph_0$-categorical and has no algebraicity over singletons, we obtain a way to produce admissible subtrees.

Proposition 2.16. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical and has no algebraicity over singletons, i.e. such that $\mathrm{acl}_{\mathfrak{M}}(a) = \{a\}$ for every $a \in \mathfrak{M}$. Let $X \subseteq \mathbf{T}^{\mathrm{eq}}$ be a finite subset. Then $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X)$ is an admissible subtree.*

More generally, if X is an admissible subtree of $\mathbf{T}^{\mathrm{eq}}$ and $Y \subseteq \mathbf{T}^{\mathrm{eq}}$ is either a finite subset or an admissible subtree, $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X \cup Y)$ is an admissible subtree.

Proof. Let X be a finite subset of $\mathbf{T}^{\mathrm{eq}}$. Since $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(\langle X\rangle)$ and $\langle X\rangle$ is finite, we may assume that X is a tree. By transitivity of $G \curvearrowright \mathbf{T}$, $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(\emptyset) = \emptyset$ and we may assume that X is non-empty. Moreover, if $X \cap \mathbf{T} = \emptyset$, then X is a singleton $\{x\}$ where $x \in \mathbf{T}^{\mathrm{eq}}\backslash\mathbf{T}$ and there exists a unique $y \in \mathbf{T}$ such that $x \sim y$. Then $G_{\{x\}} = G_{\{x,y\}}$ hence $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X \cup \{y\})$ and we can assume that $X \cap \mathbf{T} \neq \emptyset$.

Claim. *After these simplifications, we have:*

$$\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T} = \bigcup_{x \in X \cap \mathbf{T}} (\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(E_X(x)) \cap \mathfrak{M})_x \,,$$

In particular, $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T}$ *is finite.*

Proof of the claim. Since $\mathfrak{M}$ has no algebraicity on singletons, G_X assigns an infinite orbit to every vertex $y \in \mathbf{T}$ that such that $d(y,X) \geqslant 2$ (we are using the assumption that X is a tree). Thus, if $y \in \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T} \backslash X$, then $d(y,X) = 1$. Since $X \cap \mathbf{T} \neq \emptyset$, we even have $d(y, X \cap \mathbf{T}) = 1$. Moreover, for every $z \in T$ such that $z \sim x$, we have by Corollary 2.7, seeing $E_X(x)$ as a finite subset of $\mathfrak{M}^{\mathrm{eq}}$:

$$G_X \cdot z \simeq \Gamma_{E_X(x)} \cdot c(x,y)$$

Thus:

$$\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T} = \bigcup_{x \in X \cap \mathbf{T}} (\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathfrak{M})_x = \bigcup_{x \in X \cap \mathbf{T}} (\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(E_X(x)) \cap \mathfrak{M})_x$$

as claimed.

Finally, for every $x \in X \cap \mathbf{T}$, $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(E_X(x)) \cap \mathfrak{M}$ is finite by Items 1 and 2 of Proposition 2.3 hence so is $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T}$. $\square_{\text{claim}}$

By the above claim, we can assume that $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T} \subseteq X$. Let $x \in \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X) \cap \mathbf{T} = X \cap \mathbf{T}$. For every $y \sim x$ in $\mathbf{T}^{\mathrm{eq}}$, we have by Corollary 2.7:

$$G_X \cdot y \simeq \Gamma_{E_X(x)} \cdot c(x,y) \,,$$

hence $E_{\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X)}(x) = \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(E_X(x))$ is locally finite, algebraically closed and definable in $\mathfrak{M}^{\mathrm{eq}}$ by Items 1 and 2 of Proposition 2.3. We conclude with Proposition 2.12.

Suppose now X is an admissible subtree and Y is a finite subset of $\mathbf{T}^{\mathrm{eq}}$. By Lemma 2.13, there exists a finite subset $X_0 \subseteq \mathbf{T}^{\mathrm{eq}}$ such that $X = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0)$. Then $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X \cup Y) = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0 \cup Y)$ and the first part of the proof applies. Similarly, if Y is an admissible subtree, apply Lemma 2.13 again to get Y_0 finite such that $Y = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y_0)$. Then $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X \cup Y) = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0 \cup Y_0)$ and the first part of the proof applies. $\square$

The previous result also allows us to endow $\mathrm{Adm}(\mathbf{T}^{\mathrm{eq}})$ with a lattice structure. Indeed, assume $\mathfrak{M}$ is $\aleph_0$-categorical and has no algebraicity over singletons. Then, given two admissible subtrees $X, Y \subseteq \mathbf{T}^{\mathrm{eq}}$, $X \wedge Y := X \cap Y$ is admissible by Corollary 2.12 and $X \vee Y := \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X \cup Y)$ is also admissible by Proposition 2.16. Moreover, since $\mathcal{D}_{\mathcal{M}}$ is well-founded and $X \cap \mathbf{T}$ is finite if X is admissible, the local characterization of admissible trees from Corollary 2.12 shows that the lattice $(\mathrm{Adm}(\mathbf{T}^{\mathrm{eq}}), \subseteq, \wedge, \vee)$ is also well founded:

Corollary 2.17. *If* $\mathfrak{M}$ *is* $\aleph_0$*-categorical and has no algebraicity over singletons,* $(\mathrm{Adm}(\mathbf{T}^{\mathrm{eq}}), \subseteq, \wedge, \vee)$ *is a well-founded lattice.*

To finish the classification, we will need some slightly stronger hypotheses on $\mathfrak{M}$, namely primitivity as well as the following:

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

The latter hypothesis can be rephrased as follows: for every $a \in \mathfrak{M}$, Γ_a admits no proper subgroup of finite index. These are indeed stronger than what we assumed above:

Lemma 2.18. *Assume* $\mathfrak{M}$ *is primitive and such that:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

Then:

$$\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset).$$

If moreover $\mathfrak{M}$ *is* $\aleph_0$*-categorical, then* $\mathfrak{M}$ *has no algebraicity over singletons:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}}(a) = \{a\}.$$

Proof. Write Γ for $\mathrm{Aut}(\mathfrak{M})$. Let $e \in \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset)$. Fixing any $a \in \mathfrak{M}$, we have $e \in \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset) \subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a)$. Hence $\Gamma_a \leqslant \Gamma_e$ and, by primitivity, $\Gamma_e = \Gamma_a$ or $\Gamma_e = \Gamma$. If $\Gamma_e = \Gamma_a$, then $a \in \mathrm{dcl}(\emptyset)$, contradicting the transitivity hypothesis. Thus $\Gamma_e = \Gamma$ and $e \in \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset)$. The other inclusion is trivial.

For the last statement, assume $\mathfrak{M}$ is $\aleph_0$-categorical and consider the equivalence relation $\sim$ on $\mathfrak{M}$ given by $a \sim b$ if $\mathrm{acl}_{\mathfrak{M}}(a) = \mathrm{acl}_{\mathfrak{M}}(b)$. This relation is Γ-invariant hence trivial by primitivity of $\mathfrak{M}$. There are two cases:

– for every $a, b \in \mathfrak{M}$, $\mathrm{acl}_{\mathfrak{M}}\, a = \mathrm{acl}_{\mathfrak{M}}\, b$:
  Then, for every $a, b \in \mathfrak{M}$, $a \in \mathrm{acl}_{\mathfrak{M}}\, a = \mathrm{acl}_{\mathfrak{M}}\, b$ and thus $\mathrm{acl}_{\mathfrak{M}}\, b = \mathfrak{M}$. This is a contradiction to the well known fact that, in an $\aleph_0$-categorical structure, the algebraic closure of a finite set is finite.
– for every $a \neq b \in \mathfrak{M}$, $\mathrm{acl}_{\mathfrak{M}}\, a \neq \mathrm{acl}_{\mathfrak{M}}\, b$:
  Let $a, b \in \mathfrak{M}$ and suppose that $b \in \mathrm{acl}(a)$. Then $\Gamma_a \cdot b$ is finite, i.e. $\Gamma_a/(\Gamma_a \cap \Gamma_b)$ is finite. Since Γ_a has no proper finite index subgroup, $\Gamma_a \subseteq \Gamma_b$. But Γ_b is a proper subgroup of Γ by transitivity of $\mathfrak{M}$. Thus, by primitivity of $\mathfrak{M}$ again, $\Gamma_a = \Gamma_b$ and $b = a$.

□

Example 2.19. Evidently, the trivial structure with equality as sole relation, which corresponds to $\mathrm{Aut}(\mathbf{T})$, the original context of [Ol'80], satisfies the above hypothesis. More generally, any $\aleph_0$-categorical structure that admits weak elimination of imaginaries and has no algebraicity over singletons will do (primitivity is automatic in this case, see [JJ25, Cor. 3.7]). This applies in particular to the Random graph or the projectivization of the countable vector space over a finite field. More examples to come.

We now turn to the key lemma for finding induced subrepresentations, adapted from [Ol'80, Lem. 5.1]. It can be interpreted as a weak form of dissociation (compare with Lemma 2.4) for the representations of G.

Lemma 2.20. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical, primitive and satisfies:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

Let $X, Y \subseteq \mathbf{T}^{\mathrm{eq}}$ be admissible subtrees, with X non-trivial and $X \not\subseteq Y$.

There exists a proper admissible subtree $X' \subseteq X$ such that for every representation $\pi\colon G \curvearrowright \mathcal{H}$ of G, we have $p_Y p_X = p_Y p_{X'}$.

Proof. First, note that one can always replace Y with a larger admissible subtree. Indeed, suppose Y' is an admissible subtree of $\mathbf{T}^{\mathrm{eq}}$ such that $Y \subseteq Y'$. Suppose also there is $X' \subseteq X$ such that for every representation $\pi\colon G \curvearrowright \mathcal{H}$ of G, we have $p_{Y'} p_X = p_{Y'} p_{X'}$. Since $Y \subseteq Y'$ we have $p_Y p_{Y'} = p_Y$ and:

$$p_Y p_X = p_Y p_{Y'} p_X = p_Y p_{Y'} p_{X'} = p_Y p_{X'}.$$

Let $x \in X \backslash Y$ be at maximal distance from Y (admissible trees have finite diameter). Then x must be a leaf in X hence adjacent to a unique vertex $y \in X \cap \mathbf{T}$. We distinguish two cases:

– **Case 1:** $y \in Y$**.**
  Set:

$$X' = X \backslash [V_X(y) \backslash Y]\ .$$

  In other words, remove from X all vertices that are adjacent to y but don't belong to Y. All of the vertices so removed are leaves in X by maximality of $d(x, Y)$, hence X' is a tree. It is strictly contained in X since $x \notin X'$. Moreover, every $x' \in X' \backslash \{y\}$ satisfies $E_{X'}(x') = E_X(x')$. By Corollary 2.12, in order to check that X' is admissible, it suffices to check that $E_{X'}(y)$ is locally Γ-finite, definable and algebraically closed in $\mathfrak{M}^{\mathrm{eq}}$. But $E_{X'}(y) = E_X(y) \cap E_Y(y)$ is indeed locally Γ-finite, definable and algebraically closed as the intersection of two such sets.

  Claim. *Up to replacing Y with a larger admissible tree, we can assume that $X' \subseteq Y$.*

Proof of the claim. Let $Y' = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y \cup X')$. By Proposition 2.16, this is an admissible subtree of $\mathbf{T}^{\mathrm{eq}}$ that contains both Y and X'. Moreover, $V_{Y'}(y) = V_Y(y)$ thus $x \notin Y'$ and $X \not\subseteq Y'$. $\square_{\text{claim}}$

We are thus left with the following simplified situation, illustrated in Figure 3: $X \backslash Y$ is only comprised of neighbors of y. Note that the elements of $X \backslash Y$ are all leaves in X hence lie in $\mathbf{T}^{\mathrm{eq}} \backslash \mathbf{T}$ by Proposition 2.10.

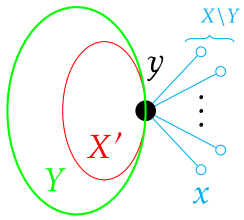

FIGURE 3. Situation after simplifications.

Using Lemma 2.13, let X_0, Y_0, X_0' be finite subtree of $\mathbf{T}^{\mathrm{eq}}$ such that

$$\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0) = X, \quad \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y_0) = Y \quad \text{and} \quad \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(X_0') = X'.$$

Up to replacing X_0, Y_0, X_0' with larger finite trees, we can assume that $X_0 \cap Y_0 = X_0'$ and that $y \in X_0'$. Let $A = E_{X_0}(y)$, $B = E_{Y_0}(y)$ and $C = E_{X_0'}(y)$, seen as finite subsets of $\mathfrak{M}^{\mathrm{eq}}$. Again, we can assume that $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A) = E_X(y)$, $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B) = E_Y(y)$ and $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(C) = E_{X'}(y)$.

Define:

$$Z = X_0' \cup (\mathfrak{M}^{\mathrm{eq}})_y = X_0 \cup (\mathfrak{M}^{\mathrm{eq}})_y \,.$$

Then $G_{X_0'}$ leaves Z, and thus $\mathcal{H}_Z$, stable. Since the action $G_{X_0'} \curvearrowright \mathcal{H}_Z$ is trivial on G_Z, and $G_{X_0'}/G_Z$ identifies to Γ_C by Lemma 2.6, restricting and factorizing π gives rise to a representation

$$\sigma \colon \Gamma_C \curvearrowright \mathcal{K} := \mathcal{H}_Z.$$

Similarly, $G_{X_0}/G_Z \simeq \Gamma_A$ and $G_{Y_0}/(G_{Y_0} \cap G_Z) \simeq \Gamma_B$, both subgroups of Γ_C. Applying Lemma 2.4 to the representation $\sigma \colon \Gamma_C \curvearrowright \mathcal{K}$ and recalling that $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A) \cap \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B) = \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(C)$ yields:

$$p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)} p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A)} = p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(C)}$$

Claim. *Projections associated with σ identify with restrictions of projections associated with π as follows:*

$$p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A)} = p^{\pi}_X\big|_{\mathcal{H}_Z}, \quad p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)} = p^{\pi}_Y\big|_{\mathcal{H}_Z} \quad \& \quad p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(C)} = p^{\pi}_{X'}\big|_{\mathcal{H}_Z}.$$

Proof of the claim. We prove it for Y only, the other identifications are similar and more straightforward since $X, X' \subseteq Z$. First, note that:

$$p^{\pi}_Y \mathcal{H}_Z = \mathcal{H}_Y \cap \mathcal{H}_Z.$$

Indeed, let $\xi \in \mathcal{H}_Z$ and $\varepsilon > 0$. By Lemma 1.11, there exists $Y_1 \subseteq Y$ finite such that:

$$\|p^{\pi}_{Y_1}\xi - p^{\pi}_Y \xi\| \leqslant \varepsilon.$$

Up to replacing Y_1 with a larger finite subset of Y, we can assume $X_0' \subseteq Y_1$. Then G_{Y_1} leaves Z, and hence $\mathcal{H}_Z$, stable. In particular, using the Alaoglu-Birkhoff Theorem 1.10, we get:

$$p^{\pi}_{Y_1}\xi \in \mathcal{H}_Z.$$

Since ε was arbitrarily small and $\mathcal{H}_Z$ is closed, $p^{\pi}_Y \xi \in \mathcal{H}_Z$. The other inclusion is trivial.

Now $p^{\pi}_Y\big|_{\mathcal{H}_Z}$ and $p^{\sigma}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)}$ are both orthogonal projectors defined on $\mathcal{H}_Z$. To prove they are equal, it suffices to show they have the same image, i.e. that $\mathcal{H}_Y \cap \mathcal{H}_Z = \mathcal{K}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)}$.

Let $\xi \in \mathcal{H}_Y \cap \mathcal{H}_Z$ and $\varepsilon > 0$. As above, there exists Y_1 a finite subset of Y containing X'_0 such that $\|p^\pi_{Y_1}\xi - \xi\| \leqslant \varepsilon$ and $p^\pi_{Y_1}\mathcal{H}_Z \subseteq \mathcal{H}_Z$. Let $B_1 = V_{Y_1}(y)$ seen as a finite subset of $\mathfrak{M}^{\mathrm{eq}}$. Let $\gamma \in \Gamma_{B_1}$. Using surjectivity in Lemma 2.6, pick $g \in G_{Y_1}$ such that $g_y = \gamma$. Then:

$$\sigma(\gamma)p^\pi_{Y_1}(\xi) = \pi(g)p^\pi_{Y_1}\xi = p^\pi_{Y_1}\xi,$$

hence $p^\pi_{Y_1}(\xi) \in \mathcal{K}_{B_1} \subseteq \mathcal{K}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)}$. Since ε was arbitrarily small and $\mathcal{K}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)}$ is closed, $\xi \in \mathcal{K}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)}$.

Conversely, let $B_1 \subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)$ be finite. Let $Y_1 = X'_0 \cup (B_1)_y$. Clearly, $\mathcal{K}_{B_1} \subseteq \mathcal{H}_Z \cap \mathcal{H}_{Y_1} \subseteq \mathcal{H}_Z \cap \mathcal{H}_Y$. Since B_1 was arbitrary and $\mathcal{H}_Z \cap \mathcal{H}_Y$ is closed, $\mathcal{K}_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)} \subseteq \mathcal{H}_Z \cap \mathcal{H}_Y$.

$\square_{\text{claim}}$

By the previous claim, and because $X, X' \subseteq Z$:

$$p^\pi_Y p^\pi_X = p^\pi_Y p^\pi_X p^\pi_Z = p^\sigma_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(B)} p^\sigma_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(A)} p^\pi_Z = p^\sigma_{\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(C)} p^\sigma_Z = p^\pi_{X'} p^\pi_Z = p^\pi_{X'},$$

which finishes the proof of the first case.

– **Case 2:** $y \notin Y$.

Let z be the unique neighbor of y in $\mathbf{T}^{\mathrm{eq}}$ such that $d(z,Y) = d(y,Y) - 1$.

Claim. *We can assume that* $z \in X$.

Proof of the claim. Indeed, suppose that $z \notin X$. By maximality of $d(x,Y)$, X must be contained in $(\mathfrak{M}^{\mathrm{eq}})_y$. More precisely, there is a definable, locally Γ-finite and algebraically closed subset $A \subseteq \mathfrak{M}^{\mathrm{eq}}$ such that $X = A_y$. By Item 2 of Proposition 2.10, we also have $A \cap \mathfrak{M} = \emptyset$. Writing $a = c(y,z)$, we distinguish two cases:

- If $A \subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a)$: then, $A = \mathrm{dcl}\,\mathfrak{M}^{\mathrm{eq}}(\emptyset)$. Indeed, since $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a)$ by hypothesis, for every $e \in A$ we have $\Gamma_a \leqslant \Gamma_e$. By primitivity of $\mathfrak{M}$ and since $A \cap \mathfrak{M} = \emptyset$, we have $\Gamma_e = \Gamma$, i.e. $e \in \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(\emptyset)$. The other inclusion is trivial. Now, $G_X = G_y$, a contradiction to X being non-trivial.
- If $A \not\subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a)$: Replace Y with $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y \cup \{y\})$ and apply Case 1.

$\square_{\text{claim}}$

Claim. *We can assume that:*

$$\forall y' \in (V_X(z) \backslash Y) \cap \mathbf{T},\ E_X(y') = \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(c(z,y')).$$

Proof of the claim. Indeed, suppose there exists $y' \in (V_X(z) \backslash Y) \cap \mathbf{T}$ such that:

$$E_X(y') \not\subseteq \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(c(z,y'))\ .$$

Replacing y with y' and x with any $x' \in V_X(y') \backslash \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(c(z,y'))$, we can assume that $y = y'$. Finally, replacing Y with $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y \cup \{y'\})$, we can apply Case 1.

$\square_{\text{claim}}$

We now assume the statements in both of the two previous claim and define X' by cutting away all the branches that emerge from z outside of Y, i.e.:

$$X' = \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}} X \backslash \bigcup_{y' \in V_X(z) \backslash Y} \mathrm{Cone}_z(y')).$$

Note that all vertices so removed are at distance at most 2 from z and that $y \notin X'$.

Similarly to the previous case, we can replace Y with $\mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y \cup X')$ to get that $X' = X \cap Y$. Finally, set:

$$Z = X \cup (\mathfrak{M}^{\mathrm{eq}})_z \cup \bigcup_{y' \in V_{\mathbf{T}}(z)} \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(c(z,y'))_{y'}\ .$$

The situation is slightly different than in Case 1. Instead of seeing the structure $\mathfrak{M}$ around y, we see a seemingly slightly larger structure around z. It can be thought of as the diameter 4 labeled subtree $\mathbf{T}_0^{\mathrm{eq}} \subseteq \mathbf{T}^{\mathrm{eq}}$:

$$(\mathfrak{M}^{\mathrm{eq}})_z \cup \bigcup_{y' \in V_{\mathbf{T}}(z)} \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(c(z,y'))_{y'} \,,$$

represented in Figure 4.

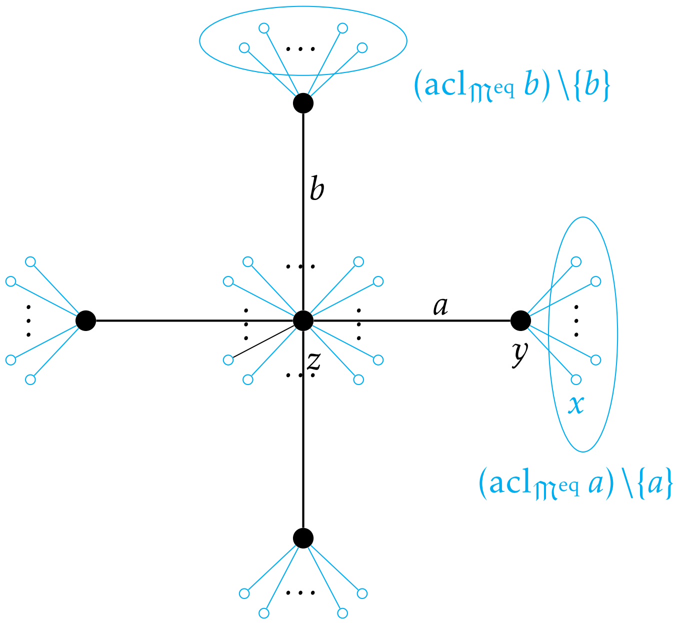

Figure 4. The larger structure.

This subtree of $\mathbf{T}^{\mathrm{eq}}$ is stable under the action of G_z and since $\mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a)$ for every $a \in \mathfrak{M}$, the permutation actions $G_z \curvearrowright \mathbf{T}_0^{\mathrm{eq}}$ and $\Gamma \curvearrowright \mathfrak{M}$ are essentially the same. The dissociation argument at the end of the proof of Case 1, invoking Lemma 2.4, applies verbatim modulo the identification of the actions $G_z \curvearrowright \mathbf{T}_0^{\mathrm{eq}}$ and $\Gamma \curvearrowright \mathfrak{M}$.

□

The previous lemma can be used to find induced representations in any representation of G with no spherical part:

Corollary 2.21. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical, primitive and satisfies:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

Let $\pi\colon G \curvearrowright \mathcal{H}$ be a non-zero unitary representation of G which has no spherical part, i.e. such that $\mathcal{H}_x = 0$ for every $x \in \mathbf{T}$. There exists a non-trivial admissible subtree $X \subseteq \mathbf{T}^{\mathrm{eq}}$ and an irreducible representation $\sigma\colon \widetilde{G}_X/G_X \curvearrowright \mathcal{K}$ such that $\mathrm{Ind}_{\widetilde{G}_X}^{G}(\sigma) \subseteq \pi$.

Proof. By Lemma 1.12, there exists a finite subtree $Y \subseteq \mathbf{T}$ such that $\mathcal{H}_Y \neq 0$. Then $Z := \mathrm{acl}_{\mathbf{T}^{\mathrm{eq}}}(Y)$ is an admissible subtree of $\mathbf{T}^{\mathrm{eq}}$ by Proposition 2.16 and Lemma 2.18 such that $\mathcal{H}_Z \supseteq \mathcal{H}_Y \neq 0$. In particular:

$$\{X \in \mathrm{Adm}(\mathbf{T}^{\mathrm{eq}}),\ \mathcal{H}_X \neq 0\} \neq \emptyset$$

and we can fix a minimal element X in this set. Since π has no spherical part, X must be non-trivial.

Claim. *For every $g,h \in G$ such that $g\widetilde{G}_X \neq h\widetilde{G}_X$, we have:*

$$\pi(g)\mathcal{H}_X \perp \pi(h)\mathcal{H}_X \,. \tag{4}$$

Proof of the claim. Up to replacing g with $h^{-1}g$, we can assume that $h = 1$. Let $\xi, \eta \in \mathcal{H}_X$. It follows:

$$(5) \qquad \langle \pi(g)\xi, \eta \rangle = \langle \pi(g)p_X\xi, p_X\eta \rangle = \left\langle p_{g(X)}\pi(g)\xi, p_X\eta \right\rangle = \left\langle \pi(g)\xi, p_{g(X)}p_X\eta \right\rangle.$$

Recalling that X is locally G-finite and $g \notin \widetilde{G}_X$, we get $X \not\subseteq g(X)$. Applying Lemma 2.20 and by minimality of X, we get $p_{g(X)}p_X = 0$. Equation (5) now gives $\langle \pi(g)\xi, \eta \rangle = 0$ and the claim follows. $\square_{\text{claim}}$

Now, consider the action of $\widetilde{G}_X$ on $\mathcal{H}_X$, which gives factorizes to a representation σ of the group $\widetilde{G}_X/G_X$. Form $\pi' := \mathrm{Ind}_{\widetilde{G}_X}^G(\sigma)$ with underlying space $\mathcal{H}'$. Let $(g_i)_{i\in I}$ be a system of coset representatives of $G/\widetilde{G}_X$. Define a map $\Phi\colon \mathcal{H}' \longrightarrow \mathcal{H}$ in the following way:

$$\forall f \in \mathcal{H}',\ \Phi(f) = \sum_{i\in I} \pi(g_i)f(g_i)\,.$$

By definition of the norm on $\mathcal{H}'$ and by Equation (4), Φ is well defined and isometric. Note that it does not depend on the choice of (g_i), for if $h^{-1}g \in \widetilde{G}_X$, then $\pi(g)f(g) = \pi(g)f(hh^{-1}g) = \pi(g)(\sigma(g^{-1}h)f(h)) = \pi(h)f(h)$. Similarly, it is easily seen that Φ is a morphism of representations.

To conclude, recall that $\widetilde{G}_X$ is Roelcke-precompact by Corollary 2.12 hence σ splits as a sum of irreducible subrepresentations by the main result of [Tsa12]. Moreover, by the basic properties of induction, if $\tau \subseteq \sigma$, then $\mathrm{Ind}_{\widetilde{G}_X}^G(\tau) \subseteq \mathrm{Ind}_{\widetilde{G}_X}^G(\sigma)$. $\square$

We are now ready to state and finish the proof of our main theorem.

Theorem 2.22. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical, primitive and satisfies:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

1. *Let X be a non-trivial admissible subtree of* $\mathbf{T}$ *and let σ be an irreducible representation of $\widetilde{G}_X$. Then* $\mathrm{Ind}_{\widetilde{G}_X}^G(\sigma)$ *is an irreducible representation of G.*
2. *If Y is another admissible subtree of* $\mathbf{T}^{\mathrm{eq}}$ *and τ is an irreducible representation of* $\mathrm{Aut}_{\mathfrak{M}}(Y)$ *such that* $\mathrm{Ind}_{\widetilde{G}_Y}^G(\tau)$ *is isomorphic to* $\mathrm{Ind}_{\widetilde{G}_X}^G(\sigma)$*, then there exists $g \in G$ such that $g(X) = Y$ and τ^g is isomorphic to σ.*
3. *Every non-spherical irreducible representation of G is of the above form.*

Proof. The only thing left to prove is Item 3. To that aim, let π be a non-spherical irreducible unitary representation of G. In particular, π is non-zero and, by Corollary 2.21, there exists a non-trivial admissible subtree X of $\mathbf{T}^{\mathrm{eq}}$ and an irreducible representation σ of $\widetilde{G}_X/G_X$ such that $\mathrm{Ind}_{\widetilde{G}_X}^G(\sigma) \subseteq \pi$. This must be an equality by irreducibility of π.

$\square$

It is worth pointing out that Corollary 2.21 combined with Zorn's Lemma also has the following consequence:

Corollary 2.23. *Assume $\mathfrak{M}$ is a primitive $\aleph_0$-categorical structure such that:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

Let π be a unitary representation of $\mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ *with no spherical part (meaning $\mathcal{H}_x = 0$ for every/some $x \in T$). Then π is dissociated and splits as a sum of non-spherical irreducible subrepresentations of the above form.*

Finally, our classification is close enough to Ol'šanskiĭ's so that his proof of Aut($\mathbf{T}$) being Type I [Ol'80, Th. 5.7] applies almost word for word, hence Corollary 1 is also proved (recall that if $\mathfrak{M}$ is 2-transitive, then $\mathfrak{M}$ is primitive).

Corollary 2.24. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical, 2-transitive and satisfies:*

$$\forall a \in \mathfrak{M},\ \mathrm{acl}_{\mathfrak{M}^{\mathrm{eq}}}(a) = \mathrm{dcl}_{\mathfrak{M}^{\mathrm{eq}}}(a).$$

Then G is of Type I.

Example 2.25. This applies in particular to 2-transitive $\aleph_0$-categorical structures that weakly eliminates imaginaries, such as the projectivization $PGL(\infty, q)$ of the countable vector space over a finite field $\mathbb{F}_q$. More examples to come.

3. The Howe-Moore property

3.1. **A model theoretical structure on T.** Let $\mathcal{L} = (\mathcal{R}_i, n_i)_{i\in I}$ be the language of $\mathfrak{M}$, **assumed to be relational**. In this notation, I is an index set and $\mathcal{R}_i$ is a relation symbol of arity n_i for every $i \in I$. Define $\mathcal{L}_{\mathbf{T}} := (\mathcal{R}_i, n_i + 1)_{i\in I} \cup \{(\sim, 2)\}$. Then $\mathbf{T}$, once $\mathfrak{M}$-colored with χ, is naturally an $\mathcal{L}_{\mathbf{T}}$-structure which satisfies $\mathrm{Aut}_{\mathcal{L}_{\mathbf{T}}}(\mathbf{T}) = G$. Indeed, interpret $\sim$ in $\mathbf{T}$ as the edge relation and, for every $i \in I$, interpret $\mathcal{R}_i$ in $\mathbf{T}$ as follows:

$$\mathcal{R}_i^{\mathbf{T}} := \{x \wedge \overline{a},\ \overline{a} \in V_{\mathbf{T}}(x)^{n_i},\ \chi(x, \overline{a}) \in \mathcal{R}_i\}.$$

This structure is still denoted $\mathbf{T}$.

In order to capture the index 2 subgroup G^+, we also define $\mathcal{L}_{\mathbf{T}}^+ := \mathcal{L}_T \cup \{(\varepsilon, 2)\}$ where ε is interpreted in $\mathbf{T}$ as:

$$\forall x, y \in \mathbf{T},\ [\varepsilon(x,y) \iff d_{\mathbf{T}}(x,y) \text{ is even}]$$

and denote this structure by $\mathbf{T}^+$.

Definition 3.1. Given a subtree $X \subseteq \mathbf{T}$ and $x \in X$, write $A(X, x) := \{c(x,y),\ y \in V_X(x)\} \subseteq \mathfrak{M}$.

Proposition 3.2. $\mathrm{Aut}_{\mathcal{L}_{\mathbf{T}}}(\mathbf{T}) = \mathrm{Aut}_{\mathfrak{M}}(\mathbf{T})$ *and* $\mathrm{Aut}_{\mathcal{L}_{\mathbf{T}}^+}(\mathbf{T}) = \mathrm{Aut}_{\mathfrak{M}}^+(\mathbf{T})$

Proof. Suppose $g \in \mathrm{Aut}_{\mathcal{L}_{\mathbf{T}}}(\mathfrak{M})$. Clearly, g is a tree automorphism. Moreover, let $x \in T$, $\overline{a} \subseteq V(x)$ and $\mathcal{R} \in \mathcal{L}$. Then:

□

Definition 3.3. Let G be a topological group. A subset A of G is said to be *coarsely bounded* if for every continuous and G-invariant écart d on G, the d-diameter of A is finite.

Definition 3.4. Let H be a topological group. A continuous map $f\colon H \longrightarrow \mathbb{C}$ *vanishes at infinity* if for every $\varepsilon > 0$, there exists a coarsely bounded set $A \subseteq H$ such that $|f(g)| \leqslant \varepsilon$ for every $g \in H\backslash A$. We will denote by $C_0(H)$ the set of such functions.

Given a sequence $(g_n)_{n\in\mathbb{N}}$ in a topological group, we will write $g_n \longrightarrow \infty$ if (g_n) eventually escapes every coarsely bounded subset of H. The geometry of our groups being nice enough, the above notion is captured by such sequences:

Lemma 3.5. *Assume $\mathfrak{M}$ is $\aleph_0$ categorical and 2-transitive. Let $f \in C(G^+)$. The following properties are equivalent:*

- *(i) f vanishes at infinity,*
- *(ii) for every $(g_n) \in G^{+\mathbb{N}}$, $g_n \longrightarrow \infty$ implies $f(g_n) \longrightarrow 0$.*

Proof. $(i) \Rightarrow (ii)$ is obviously true in higher generality. We prove the converse by contrapositive: suppose f does not satisfy (i). Then, there exists $\varepsilon > 0$ such that $N := \{g \in G,\ |f(g)| \geqslant \varepsilon\}$ is not coarsely bounded. By Corollary 1.9, $N \cdot x$ is unbounded in $\mathbf{T}$. Thus, for every $n \in \mathbb{N}$, there exists $g_n \in N$ such that $d(g_n x, x) \geqslant n$. We have $g_n \longrightarrow \infty$ but $|f(g_n)| \geqslant \varepsilon$, contradicting (ii). □

Definition 3.6. A topological group G has the *Howe-Moore property* if for every representation $\pi\colon G \curvearrowright \mathcal{H}$ with no invariant vector, all the matrix coefficients of G arising from π vanish at infinity.

Theorem 3.7. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical and 2-transitive. The group $\mathrm{Aut}_{\mathfrak{M}}^+(\mathbf{T})$ has the Howe-Moore property.*

In the following lemmas, we assume $\mathfrak{M}$ to be $\aleph_0$-categorical and 2-transitive. We also fix $\tau \in G^+$ a step 2 translation in G^+ and denote by ω^- and ω^+ the ends of its axis $\Delta =]\omega^-, \omega^+[$, assuming $\tau^n x \longrightarrow \omega^+$ as $n \longrightarrow +\infty$. The proof follows the classical one from the finite arity case. The main difference is that since our groups are not locally compact, a finer Cartan decomposition is required. Namely, we give a description of $G_A \backslash G^+ / G_B$ for every finite subtrees $A, B \subseteq \mathbf{T}$, not just $A = B = \{x\}$, seeing this double quotient as a space of types. From now on, such A and B non-empty are fixed. Let $n = |A|$, $m = |B|$ and write $A = \{a_1, \ldots, a_n\}$, $\overline{a} = (a_1, \ldots, a_n)$, $B = \{b_1, \ldots b_m\}$ and $\overline{b} = (b_1 \ldots, b_m)$.

Definition 3.8. Let $\mathcal{N}$ be a structure and let $\overline{x} = \overline{y} \wedge \overline{z}$ where $\overline{y} = (y_1, \ldots, y_n)$ and $\overline{z} = (z_1, \ldots, z_m)$ are tuple of variable. For $p = p(\overline{x}) \in \mathbf{S}_{\overline{x}}(\mathcal{N})$, we define $\pi_1(p) = p\big|_{\overline{y}} \in \mathbf{S}_{\overline{y}}(\mathcal{N})$ and $\pi_2(p) = p\big|_{\overline{z}} \in \mathbf{S}_{\overline{z}}(\mathcal{N})$

Throughout, the distance at play is the graph distance, definable using using only $\sim$, and $d(\overline{x}, \overline{y}) = \infty$ is the type-definable condition $[d(\overline{x}, \overline{y}) \geqslant n$ for every $n]$ which means that $\overline{x}$ and $\overline{y}$ are not contained in the same connected component.

Lemma 3.9. *Consider the map*

$$\Phi \colon G_A \backslash G^+ / G_B \longrightarrow \mathbf{S}_{\overline{x}}(\mathbf{T}^+), \qquad G_A^+ g G_B^+ \longmapsto \mathrm{tp}_{\mathbf{T}^+}\left(\overline{a}, g \cdot \overline{b}\right).$$

1. *Φ is a homeomorphism onto its image.*
2. $\mathrm{Im}(\Phi) = \left\{p(\overline{x}),\ \pi_1(p) = \mathrm{tp}_{\mathbf{T}^+}(\overline{a}),\ \pi_2(p) = \mathrm{tp}_{\mathbf{T}^+}\left(\overline{b}\right) \text{ and } d(\overline{y}, \overline{z}) < \infty\right\}$,
3. $\overline{\mathrm{Im}(\Phi)} = \left\{p(\overline{x}),\ \pi_1(p) = \mathrm{tp}_{\mathbf{T}^+}(\overline{a}),\ \pi_2(p) = \mathrm{tp}_{\mathbf{T}^+}\left(\overline{b}\right)\right\} = \mathrm{Im}(\Phi) \bigcup \{p_\infty\}$ *where p_∞ is the unique type in $\mathbf{S}_{\overline{x}}(\mathbf{T}^+)$ such that $\pi_1(p) = \mathrm{tp}_{\mathbf{T}^+}(\overline{a})$, $\pi_2(p) = \mathrm{tp}_{\mathbf{T}^+}\left(\overline{b}\right)$ and $p_\infty \vdash [d(\overline{y}, \overline{z}) = \infty]$.*

Given $g \in G^+$ such that $g(B) \cap A = \emptyset$, let x_A^g and x_B^g be the unique vertices of A and B respectively such that $d(A, g(B)) = d\left(x_A^g, g\left(x_B^g\right)\right)$. Let also α^g (resp. β^g) be the element of $\mathfrak{M}$ coloring the unique edge going from x_A^g toward $g(B)$ (resp. from $g\left(x_B^g\right)$ toward A). Write $t_A^g := \mathrm{tp}_{\mathfrak{M}}\left(\alpha^g \,\middle|\, A\left(x_A^g\right)\right)$ and $t_B^g := \mathrm{tp}_{\mathfrak{M}}\left(\left(g_{x_B^g}\right)^{-1} \cdot \beta^g \,\middle|\, B\left(x_B^g\right)\right)$. Finally, define:

$$\Psi(g) := \left(x_A^g, x_B^g, t_A^g, t_B^g, d\left(A, g(B)\right)\right).$$

Lemma 3.10. *Let $g, h \in G^+$ be such that $g(B) \cap A = \emptyset$ and $h(B) \cap A = \emptyset$. The following are equivalent:*

(i) $G_A g G_B = G_A h G_B$,
(ii) $\Psi(g) = \Psi(h)$.

Proof. Clearly, Ψ is left G_A-invariant and right G_B-invariant hence $(i) \Rightarrow (ii)$. Conversely, suppose $\Psi(g) = \Psi(h)$. We will write $x_A := x_A^g = x_A^h$ and $x_B := x_B^g = x_B^h$. Let C be the finite subtree of $\mathbf{T}$ generated by A and $g(B)$, i.e $C = A \cup g(B) \cup \{x_1, \ldots, x_d\}$ where $d = d(A, g(B)) - 1$, $x_1 \sim x_A$, $x_d \sim g(x_B)$ and $x_i \sim x_{i+1}$ for every $i < d$. Then, the shortest path from A to $g(B)$ is $x_A = x_0 \sim x_1 \sim \cdots \sim x_d \sim x_{d+1} = g(x_B)$. Similarly, let $x_A = y_0 \sim y_1 \sim \cdots \sim y_d \sim y_{d+1} = h(x_B)$ be the shortest path from A to $h(B)$. To carry out the proof, we will build some $u \in G_A$ such that $ug(b) = h(b)$ for every $b \in B$ by specifying local actions on C and applying Lemma 1.4. Recalling that $\aleph_0$-categorical structures are homogeneous, we proceed as follows:

- A: Since $\mathrm{tp}_{\mathfrak{M}}(\alpha^g | A(x_A)) = t_A^g = t_A^h = \mathrm{tp}_{\mathfrak{M}}\left(\alpha^h | A(x_A)\right)$, there exists $\gamma_{x_A} \in \Gamma_{A(x_A)}$ such that $\gamma_{x_A}(\alpha^g) = \alpha^h$. For every $a \in A \backslash \{x_A\}$, set $\gamma_a = \mathrm{id}_{\mathfrak{M}}$.
- $g(B)$: Similarly, since $t_B^g = t_B^h$, there exists $\gamma_{g(x_B)} \in h_{x_B} \Gamma_{B(x_B)} \left(g_{x_B}\right)^{-1}$ such that $\gamma_{g(x_B)} \cdot \beta^g = \beta^h$. For every $b \in B \backslash \{x_B\}$, set $\gamma_{g(b)} = h_b\left(g_b\right)^{-1}$.
- $[x_1, x_d]$: By 2-transitivity of $\mathfrak{M}$, we can find for every $i \in \{1, \ldots, d\}$ some $\gamma_{x_i} \in \Gamma$ such that $\gamma_{x_i} \cdot c(x_i, x_{i+1}) = c(y_i, y_{i+1})$ and $\gamma_{x_i} \cdot c(x_i, x_{i-1}) = c(y_i, y_{i-1})$.

Then $(\gamma_c)_{c \in C}$ satisfies the conditions of Lemma 1.4 hence gives $u \in G$ such that $u_c = \gamma_c$ for every $c \in C$ and $u(x_A) = x_A$. By construction, $u \in G_A$ and $ug(b) = h(b)$ for every $b \in B$. We have thus shown that $h \in G_A g G_B$ i.e. $G_A g G_B = G_A h G_B$. The situation is represented in Figure 5.

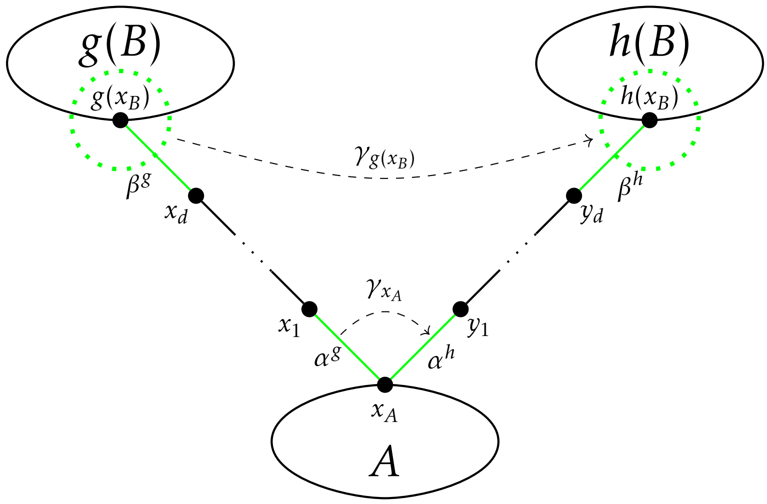

Figure 5. Local actions at x_A and $g(x_B)$. Green edges are sent to green edges. At the bottom, γ_{x_A} fixes $A(x_A)$ and sends α^g to α^h. Up above, $\gamma_{g(x_B)}$ reverses the action of g_{x_B} on $B(x_B)$ and replaces it with that of h_{x_B} while sending β^g to β^h. The dotted circles represent $V_{gB}(g(b))$ and $V_{hB}(h(b))$ respectively.

□

Note that the first four components of Ψ take values in a finite set (recall that $\mathfrak{M}$ is assumed to be $\aleph_0$-categorical).

Proof. □

Lemma 3.11. *There exists $F \subseteq G^+$ finite such that:*

$$G^+ = G_A F \{\tau^n,\ n \geqslant 0\} F G_B.$$

Proof. Let $K := \{g \in G^+,\ g(B) \cap A \neq \emptyset\}$. Then K is Roelcke-precompact in G^+ by Lemma 1.8. Thus the existence of $F_0 \subseteq K \subseteq G^+$ such that $K \subseteq G_A F_0 G_B$. It remains to deal with $G^+ \backslash K$.

With an eye towards Lemma 3.10, fix $a \in A,\ b \in B, t_A \in \mathbf{S}_1(A(a)),\ t_B \in S_1(B(b))$ where we assume t_A and t_B are realized by some $\alpha \in \mathfrak{M}\backslash A(a)$ and $\beta \in \mathfrak{M}\backslash B(b)$ respectively. Let also $\delta = 0$ if $d(a,b)$ is even or $\delta = 1$ if $d(a,b)$ is odd. We will produce $g = g(b,t_B)$ and $h = h(a,t_A)$ in G^+ such that for every $n \in \mathbb{N}$,

$$\Psi(h\tau^n g) = (a, b, t_A, t_B, 2(n+1) - \delta).$$

Note that for every $g \in G^+$, we have $d(a, g(b)) \mod 2 = d(a,b) \mod 2$ hence this will be enough to conclude, after setting

$$F := F_0 \cup \bigcup_{a,b,t_A,t_B} \{g(b,t_B), h(a,t_A)\}.$$

Let c be the unique neighbor of b such that $\chi(b,c) = \beta$. Write $\Delta = \{x_i,\ i \in \mathbb{Z}\}$ in such a way that $\tau(x_i) = x_{i+2}$ for every $i \in \mathbb{Z}$ and $d(x_0, b)$ is even. By Item 3 of Lemma 1.7, $G \curvearrowright \mathbf{T}$ is 2-transitive and we can find $g \in G$ such that $g(b) = x_0$ and $g(c) = x_{-1}$. Note that since $d(g(b), b) = d(x_0, b)$ is even, $g \in G^+$.

Next, let d be the unique neighbor of a such that $\chi(a,d) = \alpha$, let d' be any neighbor of d distinct from a and fix Δ' a bi-infinite geodesic in $\mathbf{T}$ such that $\Delta' \cap A = \{a\}$ and $d, d' \in \Delta$. Using 2-transitivity of $\mathfrak{M}$ and Lemma 1.4, one can find $h \in G$ such that:

- $h(\Delta) = \Delta'$,

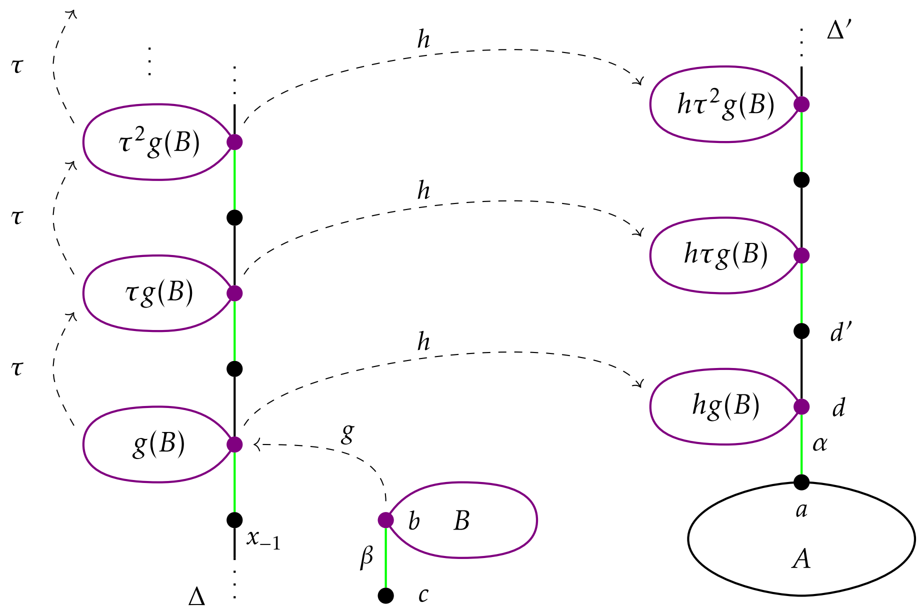

FIGURE 6. Realization of a given value of Ψ. Follow the dotted arrows to track where B and c are sent. Green edges are sent to green edges.

– $(h(x_{-1}), h(x_0)) = \begin{cases} (a,d) & \text{if } d(x_0,d) \text{ is even,} \\ (d,d') & \text{otherwise.} \end{cases}$

Again, by parity of $d(h(x_0), x_0)$, we in fact have $h \in G^+$. The situation is represented in Figure 6 (in the case $d(x_0,d)$ even). Because for every $n \in \mathbb{N}$, no edge of $h\tau^n g(B)$ ever belongs to Δ', we indeed have:

$$\forall n \in \mathbb{N},\ \Psi(h\tau^n g) = (x_A, x_B, t_A, t_B, 2(n+1) - \delta)\,.$$

□

Just as in the finite arity case, we have reduced the problem to studying the vanishing matrix coefficients of G on the abelian semigroup $\{\tau^n,\ n \in \mathbb{N}\}$:

Corollary 3.12. *Let $\pi\colon G^+ \curvearrowright \mathcal{H}$ be a representation of G^+ and suppose there exists $\xi, \eta \in \mathcal{H}$ such that $f^\pi_{\xi,\eta}$ does not vanish at infinity. Then, there exists $\xi', \eta' \in \mathcal{H}$ such that:*

$$f^\pi_{\xi',\eta'}(\tau^n) \underset{n\to+\infty}{\not\longrightarrow} 0\,.$$

Proof. Suppose $f^\pi_{\xi,\eta}$ does not vanish at infinity. Approximating ξ and η by invariant vectors using Lemma 1.12 yields a uniform approximation of f. We can thus assume $\xi, \eta \in \mathcal{H}_A$ for some non-empty finite subtree $A \subseteq \mathbf{T}$. By Lemma 3.5, there exists $\varepsilon > 0$ and $(g_n) \in G^{+\mathbb{N}}$ such that $g_n \longrightarrow \infty$ while $\left|f^\pi_{\xi,\eta}(g_n)\right| \geqslant \varepsilon$ for every $n \in \mathbb{N}$. By Lemma 3.11, we have $G^+ = G_A F\{\tau^n,\ n \in \mathbb{N}\} F G_A$ for some finite subset $F \subseteq G^+$ and we can write $g_n = u_n f_n \tau^{k_n} f'_n u'_n$ for some $u_n, u'_n \in G_A$, $f_n, f'_n \in F$ and $k_n \in \mathbb{N}$. Up to extraction, we can assume that (f_n) and (f'_n) are constant sequences and denote their respective value by f and f'.

We claim that $k_n \longrightarrow +\infty$. Indeed, G_A and F are coarsely bounded by Corollary 1.9, hence $\tau^{k_n} \longrightarrow \infty$. Again, since finite sets are coarsely bounded, we necessarily have $k_n \longrightarrow +\infty$.

Set $\xi' := \pi(f')\xi$ and $\eta' := \pi(f^{-1})\eta$. Then, for all $n \in \mathbb{N}$:

$$f^\pi_{\xi,\eta}(g_n) = f(u_n f \tau^{k_n} f' u'_n) = \left\langle \pi(\tau^{k_n})\pi(f')\pi(u'_n)\xi, \pi(f^{-1})\pi(u_n^{-1})\eta \right\rangle = \left\langle \pi(\tau^{k_n})\xi', \eta' \right\rangle = f^\pi_{\xi',\eta'}(\tau^{k_n}).$$

We thus have $\left|f^{\pi}_{\xi',\eta'}(\tau^{k_n})\right| \geqslant \varepsilon$ for every $n \in \mathbb{N}$ while $k_n \longrightarrow +\infty$. In particular, $f^{\pi}_{\xi',\eta'}(\tau^n) \not\longrightarrow 0$. □

The classical proof of Lubotzky and Mozes now carries through verbatim. More precisely, Propositions 1 and 2 in [LM92] hold for G^+. Once combined, they yield the following, which finishes the proof of Theorem 3.7:

Proposition 3.13. *Let $\pi\colon G^+ \curvearrowright \mathcal{H}$ be a representation of G^+ with no G^+-invariant vector. Then $f^{\pi}_{\xi,\eta}(\tau^n) \longrightarrow 0$ for every $\xi, \eta \in \mathcal{H}$.*

Proof. See Propositions 1 and 2 in [LM92]. □

3.2. **Model theory.** Want to describe the theory of $\mathbf{T}$ and (countable) models of it. Given a tuple $\overline{a}$ in $\mathbf{T}$ and a vertex $c \in \mathbf{T}$, $c \sim \overline{a}$ will mean $c \sim a_i$ for every $i \leqslant n$. We also define localized quantifiers, to be used in building $\mathcal{L}^{\mathbf{T}}_{\mathfrak{M}}$ formulae.

Definition 3.14.

1. Let x_0 be a variable. The *localized existential quantifier* is $\exists y \sim x_0, := \exists y, y \sim \overline{x} \wedge$. Similarly, the *localized universal quantifier* is $\forall y \sim \overline{x}, := \forall y, y \sim \overline{x} \rightarrow$.
2. Given $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{M}}$, we define the formula $\varphi_{\mathbf{T}}(x_0, \overline{x}) \in \mathcal{L}^{\mathbf{T}}_{\mathfrak{M}}$ by replacing every quantifier by its localized version, always using the same variable x_0, and inputting x_0 as the extra variable in every relation symbol.

Example 3.15. If $\varphi = \forall x, R(x)$ where R is a relation in $\mathcal{L}_{\mathfrak{M}}$, then $\varphi_{\mathbf{T}}(x_0)$ is the $\mathcal{L}^{\mathbf{T}}_{\mathfrak{M}}$ formula $\forall x \sim x_0, R(x_0, x) = \forall x_0,\ x \sim x_0 \rightarrow R(x_0, x)$. More generally, if $\varphi = \forall x_1 \exists x_2 \ldots \bigwedge_i \bigvee_j R_{i,j}(\overline{x_{i,j}})$, then $\varphi_{\mathbf{T}}(x_0) = \forall x_1 \sim x_0 \exists x_2 \sim x_0 \ldots \bigwedge_i \bigvee_j R_{i,j}(\overline{x_0, x_{i,j}})$. Note that under our assumptions on $\mathfrak{M}$, every formula in $\mathcal{L}_{\mathfrak{M}}$ is equivalent up to $\mathrm{Th}(\mathfrak{M})$ to one of the above form, i.e. a positive prenex formula.

Proposition 3.16. *Let $\varphi(\overline{x}) \in \mathcal{L}$ be a positive formula. For every $a_0 \in \mathbf{T}$ and $\overline{a} \in \mathbf{T}^{\overline{x}}$ such that $a_0 \sim \overline{a}$, the following are equivalent:*

(i) $\mathbf{T} \models \varphi_{\mathbf{T}}(a_0, \overline{a})$,
(ii) $\mathfrak{M} \models \varphi(\chi(a_0, \overline{a}))$.

In particular, if φ is a positive sentence, $\mathfrak{M} \models \varphi$ if and only if $\mathbf{T} \models \forall x, \varphi_{\mathbf{T}}(x)$.

Proof. We prove it by induction on the complexity of φ. If φ is atomic, this is clear by definition of the $\mathcal{L}_{\mathfrak{M}}$-structure $\mathbf{T}$. Stability under boolean combinations of formulae is also trivial. □

Definition 3.17. Let

$$\mathcal{T}^{\mathfrak{M}}_{\infty} := \mathrm{Th}_{\{\sim\}}(\mathbf{T}) \cup \bigcup_{\varphi \in \mathrm{Th}_{\mathcal{L}_{\mathfrak{M}}}(\mathfrak{M})} \{\forall x\ \varphi_{\mathbf{T}}(x)\ ,\ \forall x \forall y\ [x \sim y \Rightarrow [\varphi_{\mathbf{T}}(x,y) \leftrightarrow \varphi_{\mathbf{T}}(y,x)]]\}.$$

Definition 3.18. Let $\mathcal{T}^{\mathfrak{M}}_{\infty}$ be the union of the following sets of formulae:

- $\mathcal{T}_{\infty}$ the theory of $\mathbf{T}$ as a graph. It can be reduced to axioms expressing that $\mathbf{T}$ is connected, acyclic and has infinitely many edges around every vertex.
- $\{\ \forall x\ \varphi_{\mathbf{T}}(x)\ ,\ \varphi \in \mathrm{Th}_{\mathcal{L}_{\mathfrak{M}}}(\mathfrak{M})\}$, expressing that we see a model of $\mathfrak{M}$ around each vertex,
- $\{\ \forall x \forall y\ [x \sim y \Rightarrow [\varphi_{\mathbf{T}}(x,y) \leftrightarrow \varphi_{\mathbf{T}}(y,x)]]\ ,\ \varphi(x) \in \mathcal{L}_{\mathfrak{M}}\}$, ensuring that copies of $\mathfrak{M}$ linked by an edge agree on the $\mathcal{L}_{\mathfrak{M}}$-type of the edge.

Proposition 3.19. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical. Let $\mathcal{M}$ be a countable model of $\mathcal{T}^{\mathfrak{M}}_{\infty}$. As a graph, $\mathcal{M}$ is an everywhere infinite forest, i.e. a disjoint union of the graph $\mathbf{T}$. Moreover, there exists $\chi_{\mathfrak{M}}$ an $\mathfrak{M}$-coloring of $\mathcal{M}$ such that for every $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{M}}$ and every $a, \overline{b} \in \mathcal{M}$ such that $a \sim \overline{b}$, denoting by $\mathcal{N}$ the connected component of a,*

$$\mathcal{M} \models \varphi_{\mathbf{T}}(a, \overline{b}) \iff \mathcal{N} \models \varphi_{\mathbf{T}}(a, \overline{b}) \iff \mathfrak{M} \models \phi(\chi(a, \overline{b}))\ .$$

Proof. In particular, $\mathcal{M}$ is a model of $\mathcal{T}_\infty$ i.e. a locally infinite graph such that every connected component is isomorphic to $\mathbf{T}$ as a graph. Fixing $a \in \mathcal{M}$, we can view $\mathcal{M}_a := V_M(a)$ as an $\mathcal{L}_{\mathfrak{M}}$ structure. Indeed, every relation symbol R of arity n in $\mathcal{L}_{\mathfrak{M}}$ can be interpreted as $\{\overline{b} \in \mathcal{M}_a,\ \mathcal{M} \vDash R(a,\overline{b})\}$. By construction, we have for every $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{M}}$ and $\overline{b} \in \mathcal{M}_a$:

$$\mathcal{M}_a \vDash \varphi(\overline{b}) \qquad \Longleftrightarrow \qquad \mathcal{M} \vDash \varphi_{\mathbf{T}}(a,\overline{b}). \tag{6}$$

Thus, in view of the second item of the definition of $\mathcal{T}_\infty^{\mathfrak{M}}$, we see that $\mathrm{Th}_{\mathcal{L}_{\mathfrak{M}}}(\mathcal{M}_a) = Th_{\mathcal{L}_{\mathfrak{M}}}(\mathfrak{M})$. Since $\mathfrak{M}$ is $\aleph_0$-categorical, $\mathcal{M}_a$ is isomorphic to $\mathfrak{M}$ and we can fix an isomorphism γ_a.

In turn, given $b \in \mathcal{M}$ such that $b \sim a$, Equation (6) and the second item in the definition of $\mathcal{T}_\infty^{\mathfrak{M}}$ ensure that $\mathrm{tp}_{\mathcal{M}_a}(b) = \mathrm{tp}_{\mathcal{M}_b}(a)$. Since $\aleph_0$-categorical structures are homogeneous (REF XXX), we can pick $\gamma_b \colon \mathcal{M}_b \longrightarrow \mathfrak{M}$ an isomorphism of $\mathcal{L}_{\mathfrak{M}}$-structures such that $\gamma_b(a) = \gamma_a(b)$.

Repeating this process inductively on the connected component of a and then separately on every connected component of $\mathcal{M}$, we obtain a family $(\gamma_a)_{a\in\mathcal{M}}$ such that:

1. For every $a \in \mathcal{M}$, γ_a is an isomorphism of $\mathcal{L}_{\mathfrak{M}}$-structures $\mathcal{M}_a \longrightarrow \mathfrak{M}$,
2. For every $a, b \in \mathcal{M}$ such that $a \sim b$, we have $\gamma_a(b) = \gamma_b(a)$.

Now, define $\chi_{\mathcal{M}}$ on the edges of $\mathfrak{M}$ by setting:

$$\chi_{\mathcal{M}}(a,b) := \gamma_a(b) \text{ for every } a,b \in \mathcal{M} \text{ such that } a \sim b\ .$$

Then $\chi_{\mathfrak{M}}$ is an $\mathfrak{M}$-coloring of every connected component of $\mathcal{M}$ such that for every $\varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{M}}$ and every $a, \overline{b} \in \mathcal{M}$ such that $a \sim \overline{b}$, denoting by $\mathcal{N}$ the connected component of a,

$$\mathcal{M} \vDash \varphi_{\mathbf{T}}(a,\overline{b}) \qquad \Longleftrightarrow \qquad \mathcal{N} \vDash \varphi_{\mathbf{T}}(a,\overline{b}) \qquad \Longleftrightarrow \qquad \mathfrak{M} \vDash \phi(\chi(a,\overline{b})\ .$$

□

Corollary 3.20. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical. The countable models of $\mathcal{T}_\infty^{\mathfrak{M}}$ are exactly the countable disjoint unions of copies of* $\mathbf{T}$. *More precisely, a model $\mathcal{M}$ of $\mathcal{T}_\infty^{\mathfrak{M}}$ is a locally infinite graph such that every connected component of $\mathcal{M}$ is isomorphic to* $\mathbf{T}$ *as an $\mathcal{L}_{\mathfrak{M}}$-structure.*

Definition 3.21.

1. A type in $p \in \mathbf{S}_{\overline{x}}(\mathcal{T}_\infty)$ *encodes a tree* if every realization of p is connected.
2. Given $p \in \mathbf{S}_{\overline{x}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$, we will denote by $p_\sim$ the restriction of p to the language of graphs and say that p *encodes a tree* if $p_\sim$ does.

Lemma 3.22. *Let $p \in \mathbf{S}_{\overline{x}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$ and write $\overline{x} = (x_1, \ldots, x_n)$. If p encodes a tree, then $p_\sim$ and the set $\{\varphi_{\mathbf{T}}(x_i,\overline{x}),\quad \varphi(\overline{x}) \in \mathcal{L}_{\mathfrak{M}},\ i \leqslant n\} \cap p$ determine p.*

Proof. □

Proposition 3.23. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical.*

1. *$\mathcal{T}_\infty^{\mathfrak{M}}$ is complete.*
2. $\mathbf{T}$ *is the prime model of $\mathcal{T}_\infty^{\mathfrak{M}}$.*
3. *The countable models of $\mathcal{T}_\infty^{\mathfrak{M}}$ are exactly the countable disjoint unions of copies of* $\mathbf{T}$. *More precisely, a model $\mathcal{M}$ of $\mathcal{T}_\infty^{\mathfrak{M}}$ is a locally infinite graph such that every connected component is isomorphic to* $\mathbf{T}$ *(as an $\mathcal{L}_{\mathfrak{M}}$-structure). In this context, each union of components of $\mathcal{M}$ is an elementary substructure of $\mathcal{M}$.*

Note that, given $n \in \mathbb{N}$, there exists a formula $\varphi_n(\overline{x})$ in the language of graphs that says "$\overline{x}$ has diameter at most n". Abuse of notation: different arity = different formulae. We will say that a type $p \in \mathbf{S}_{\overline{x}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$ has bounded diameter if $\varphi_n \in p$ for some $n \in \mathbb{N}$. For $\overline{y}, \overline{z} \subseteq \overline{x}$, we will write $(d(\overline{y},\overline{z}) = \infty) \in p$ if $(d(\overline{x},\overline{y}) \geqslant n) \in p$ for every n (the latter can be expressed as a legitimate formula in the language of graphs).

Lemma 3.24. *Assume $\mathfrak{M}$ is $\aleph_0$-categorical and let $p \in \mathbf{S}_{\overline{x}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$. Then p is isolated if and only if it has bounded diameter.*

Proof. Suppose p is isolated. Then p is realized in the prime model $\mathbf{T}$. Since $\mathbf{T}$ is connected and p has finitely many variable, p has bounded diameter.

Conversely, suppose p has bounded diameter. There exists a countable model of $\mathcal{T}_\infty^{\mathfrak{M}}$ that realizes p, say $\mathcal{M}$ and $\overline{a} \in \mathcal{M}^{\overline{x}}$. By Proposition 3.19, $\mathcal{M}$ is a countable unions of copies of $\mathbf{T}$, each copy being an elementary substructure by **??**. Since p has finite diameter, $\overline{a}$ is entirely contained in a single copy of $\mathbf{T}$. Thus p is realized in the prime model of $\mathcal{T}_\infty^{\mathfrak{M}}$ (Item 2 of Proposition **??**) hence must be isolated [TZ12, Th. 4.5.3]. $\square$

Corollary 3.25. *Let $p \in \mathbf{S}_{\overline{x}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$. Up to reordering $\overline{x}$, there exists a unique partition $\overline{x} = \overline{x(1)} \wedge \cdots \wedge \overline{x(n)}$ such that $p_i := p\big|_{\overline{x(i)}}$ is isolated for every $i \le k$ and $(d(\overline{x(i)}, \overline{x(j)}) = \infty) \in p$ for every $i \neq j$.*

Conversely, for every partition $\overline{x(1)} \wedge \cdots \wedge \overline{x(n)}$ of $\overline{x}$ and for all isolated types $p_1 \in \mathbf{S}_{\overline{x(1)}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right), \dots, p_k \in \mathbf{S}_{\overline{x(k)}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$, there exists a unique $p \in \mathbf{S}_{\overline{x}}\left(\mathcal{T}_\infty^{\mathfrak{M}}\right)$ that decomposes accordingly.

Proof. To decompose p, realize it in a countable model of $\mathcal{T}_\infty^{\mathfrak{M}}$, say by $\mathcal{M}$ and $\overline{a} \in \mathcal{M}^{\overline{x}}$. By Item 3 of Proposition 3.19, $\mathcal{M}$ is a countable unions of copies of $\mathbf{T}$. Partition $\overline{x} = \overline{x(1)} \wedge \cdots \wedge \overline{x(n)}$ depending on which connected component each element of $\overline{a}$ falls into. In this decomposition, p_i has bounded diameter for every $i \leqslant k$, hence is isolated by Lemma 3.24, and $(d(\overline{x(i)}, \overline{x(j)}) = \infty) \in p$ for every $i \neq j$. This decomposition is clearly unique.

Conversely, suppose a decomposition $x = \overline{x(1)} \wedge \cdots \wedge \overline{x(n)}$ and isolated types $p_1, \dots, p_k$ are given. Recalling that isolated types are realized in every model, we can realize each p_i in a distinct copy $\mathcal{M}_i$ of $\mathbf{T}$, say by $\overline{a(i)} \in \mathcal{M}_i^{\overline{x(i)}}$. Then $\mathcal{M} := \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k$ is a model $\mathcal{T}_\infty^{\mathfrak{M}}$ and even an elementary extension of each of the $\mathcal{M}_i$ by Item 3 of Proposition 3.19. Thus $p := \mathrm{tp}_{\mathcal{M}}(\overline{a(1)} \wedge \cdots \wedge \overline{a(k)})$ admits the desired decomposition. $\square$

Corollary 3.26.

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Universite Claude Bernard Lyon 1, ICJ UMR5208, CNRS, Centrale Lyon, INSA Lyon, Université Jean Monnet, 69622 Villeurbanne, France.

Email address: barritault@math.univ-lyon1.fr